Hamilton-Jacobi formalism for gauge theories

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Abstract: The main goal of this work is to study the Hamilton-Jacobi formalism for singular systems (or systems with gauge symmetries). We review the problem that exists with these systems to understand the motivation to construct what it is known as Dirac Hamiltonian or total Hamiltonian for the dynamics of constrained systems. We conclude with the study of the free particle in the Minkowski spacetime as example.

I. INTRODUCTION

In the fifties, Peter Gabriel Bergmann with different collaborators (Bergmann 1949, Bergmann and Brunings 1949, Anderson and Bergmann 1951) and Paul Adrien Maurice Dirac (1950), working separately, started to study the canonical formalism for gauge theories. This formalism, nowadays, is also known as constrained systems.

The main objective of this work is to discuss the problems that singular systems present, to see the solution that Dirac and Bergmann proposed (the total Hamiltonian) and finally to contemplate different aspects of the Hamilton-Jacobi formalism for these systems. To conclude, we introduce an example of singular system: the free particle in the Minkowski spacetime, the simplest example to put into practice the solution given by Dirac and Bergmann. Thanks to this example, we will understand what it is called gauge symmetries. Finally, we will obtain its Hamilton-Jacobi equation.

In Section 2 we begin with a brief introduction to the Constrained Systems. In Section 3, we introduce the dynamic for constrained systems (The total Hamiltonian). In Section 4, it is discussed the Hamilton-Jacobi theory for these systems and finally, in Section 5, we study the example of the free particle.

II. DIRAC-BERGMANN CONSTRAINED SYSTEMS

Our starting point will be a time-independent Lagrangian, $L(q,\dot{q})$, defined in configuration-velocity space, TQ, that is the tangent space of manifold Q of dimension n.

As we already know, to obtain the equations of motion in the language of the mechanics, we always use the Lagrange equations,

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial \dot{x}^i \partial x^j}\dot{x}^j - \frac{\partial L}{\partial \dot{x}^i \partial \dot{x}^j}\ddot{x}^j = 0 \quad (1)$$

where $i,j = 1,2,...,n$. For regular systems what is satisfied is that the Hessian matrix

$$W_{ij} \equiv \frac{\partial L}{\partial \dot{x}^i \partial \dot{x}^j} \quad (2)$$

has determinant different from 0 and therefore, we can always isolate $\ddot{x}^i = f^i(x,\dot{x})$. In addition, the theorem about the existence and uniqueness of solutions of ordinary differential equation guarantees us the existence of the solution and, moreover, this solution is unique given the initial conditions.

However, for singular systems or constrained systems, the determinant of eq.(2) is equal to 0, (this is the reason why they are called singular systems) and as a consequence of this, we can not apply the theorem about existence and uniqueness of solutions. The second important consequence of the matrix being singular, it is that we have problems to construct the canonical formalism. The Legendre transformation from the configuration-velocity space TQ to the phase space T*Q ($\hat{p}(q,\dot{q}) \equiv \partial L/\partial \dot{q}$),

$$\begin{aligned} FL:\ & TQ \ \rightarrow\ T^*Q \\ & (q,\dot{q}) \ \rightarrow\ (q,\hat{p}) \end{aligned} \quad (3)$$

is not invertible because of the determinant of Hessian matrix ($\det(\partial\hat{p}/\partial\dot{q}^i) = det(\partial^2 L/\partial\dot{x}^i\partial\dot{q}^j) = 0$). Then it appears a problem with the projectability of structures from configuration space TQ to phase space T^*Q. Some structures (tensors, forms, vector fields, etc.) which are defined in TQ can not be projected to the phase space. Therefore, we need to find the projection conditions.

First of all, we are going to define the primary constraints. The primary constraints, ϕ_μ, form a basis of a set of function whose pull-back to the tangent space is equal to zero, in other words,

$$(FL^*\phi_\mu)(q,\dot{q}) = \phi_\mu(q,\hat{p}) = 0 \qquad \forall q,\dot{q} \quad (4)$$

where $\mu = 1,2,....,k_0$ (k is the number of independent primary contraints). In fact, applying $\frac{\partial}{\partial \dot{q}}$ to (4) we can get the basis of null vectors for the Hessian,

$$W_{ij}\left(\frac{\partial \phi_\mu}{\partial p_j}\right)\Bigg|_{p=\hat{p}} = 0 \qquad \forall q,\dot{q}. \quad (5)$$

With this result, we already know that the image of TQ in T^*Q is given by the primary constraints' surface of dimension $2n-k_0$. A foliation in TQ of dimension k_0 is also defined with each element given as the inverse image of a point in the primary constraints' surface T^*Q. These vector fields tangent to the surfaces of the of the foliation are generated by,

$$\Gamma_\mu = \left(\frac{\partial \phi_\mu}{\partial p_j}\right)\Bigg|_{p=\hat{p}} \frac{\partial}{\partial \hat{q}^j}. \quad (6)$$

As a result, if we define a real-valued function $f^L : TQ \to \mathbb{R}$ and $f^H : T^*Q \to \mathbb{R}$, we are able to define the projectability condition of the following way,

$$\Gamma_\mu f^L = 0, \mu = 1, 2, ..., k_0 \Longleftrightarrow \exists f^H | FL^* f^H = f^L. \quad (7)$$

In general, the dynamics may generate other, secondary, constraints. A constraint is a function that must vanish as a consequence of the equations of motion (EOM). A first class constraint, in addition, has its Poisson bracket with any other constraints, vanishing on the constraint surface. The rest, they are called second class.

III. THE DYNAMICS FOR CONSTRAINED SYSTEMS

An important part of the canonical formalism is the Hamiltonian function. As we know, the Hamiltonian in regular systems is the vector field that generates the dynamic in the phase space, $\{-, H\}$, where $\{-, -\}$ is the Poisson bracket. We can still work with it because the energy satisfies the projectability conditions ($\Gamma_\mu E = 0$ where $E = \dot{q}(\partial L/\partial \dot{q}) - L$), so we can define H_c, called canonical Hamiltonian, as the function in the phase space whose pull-back is the energy, $FL^* H_c = E$.

There is an ambiguity in the definition of H_c. As we can see, if we introduce $H_c + v^\mu \phi_\mu$, where $v^\mu(\tau)$ is an arbitrary function, it also complies with the condition. This little modification has a huge importance in the physics: this is the door leading to gauge freedom.

It is defined as total Hamiltonian or Dirac's Hamiltonian $H_D = H_c + v^\mu \phi_\mu$. In the regular case, EOM are written of the following way,

$$\dot{q} = \{q, H_c\} \;\; ; \;\; \dot{p} = \{p, H_c\} \quad (8)$$

but now, we have to introduce the new term, so the EQM are

$$\begin{gathered} \dot{q} = \{q, H_D\} = \{q, H_c\} + v^\mu \{q, \phi_\mu\} \\ \dot{p} = \{p, H_D\} = \{p, H_c\} + v^\mu \{p, \phi_\mu\} \\ \phi_\mu(q, p) = 0 \end{gathered} \quad (9)$$

out of which, new (secondary) constraints may appear.

IV. HAMILTON-JACOBI FORMALISM

We are going to consider the following action integral

$$S(t, x; t_0, x_0) = \int_{t_0}^{t} L(x(\tau)) d\tau \quad (10)$$

where $x(\tau)$ is the solution that only depends on the initial and final condition, x_0 and x respectively. If we calculate $\partial S/\partial x^i$, it is easy to find that

$$\frac{\partial S}{\partial x^i} = \frac{\partial L}{\partial \dot{x}^i}\Bigg|_{\tau=t} = p_i. \quad (11)$$

Knowing this, we are going to show that $\partial S/\partial t = -H$. From the definition of the action, its total time derivative along the path is $dS/dt = L$. Regarding S as a function of co-ordinates and time, and using eq.(11), we have

$$\frac{dS}{dt} = L = \frac{\partial S}{\partial t} + \sum\nolimits_i \frac{\partial S}{\partial x_i}\dot{x}_i = \frac{\partial S}{\partial t} + \sum\nolimits_i p_i \dot{x}_i \quad (12)$$

and therefore, $\partial S/\partial t = L - \sum p_i \dot{x}_i$ or

$$\frac{\partial S}{\partial t} = -H \quad (13)$$

We are ready to introduce the Hamilton-Jacobi equation considering that we know the theory of canonical formalism. If we take the relation (13), and we expand it to n variables and replace $p = \partial S/\partial x$, we obtain

$$\frac{\partial S}{\partial t} + H(x_1, ..., x_n; \frac{\partial S}{\partial x_1}, ..., \frac{\partial S}{\partial x_n}) = 0 \quad (14)$$

what it is known as Hamilton-Jacobi equation. Then, the solution will be,

$$S = S(t, x_1, ..., x_n; \alpha_1, ..., \alpha_n) + A \quad (15)$$

where $\alpha_1, ..., \alpha_n$ (or $x_0^1, ..., x_0^n$) and A are arbitrary constants. It seems that we can have n arbitrary constants (the α's), the initial time, t_0, and one additive constant, A. So, as we can see, we have n+2 constants. However, this is not true. We can always separate in two parts the trajectory, $S(t, x; t_0, x_0) = S_2(t, x; t_1, x_1) + S_1(t_1, x_1; t_0, x_0)$. As a consequence of this, S_1 is an additive constant (basically because t_1, x_1, t_0, x_0 are fixed parameters in our trajectory). The decision that we take is to use it to move the initial time, t_0, to 0. Therefore, really we have $n+1$ independent parameters, one of them additive. Now, we take $S(t, q; \alpha)$ as generating function and α_i as the new parameter for the complete solution. We can use S to make a canonical transformation, so that α_i become the new co-ordinates and $\beta_i = -(\partial S/\partial \alpha_i)$ will be the new momenta. Thus, the new Hamiltonian is,

$$H' = H + \frac{\partial S}{\partial t} \quad (16)$$

which vanishes because of (13). Therefore, we take the Hamiltonian equation for H',

$$\begin{gathered} \dot{\alpha} = 0 \;\to\; \alpha = constant \\ \dot{\beta} = 0 \;\to\; \beta = constant \end{gathered} \quad (17)$$

Now, for gauges theories whose constraints are first class, the Hamilton-Jacobi equations are

$$\begin{gathered} \frac{\partial S}{\partial t} + H_c = 0 \\ \phi_\mu(q, \frac{\partial S}{\partial x}) = 0 \end{gathered} \quad (18)$$

where ϕ_μ are all first class constraints. We note that we have k arbitrary constants less due to the number of independent first class constraints.

Before starting with the example of the free particle, we should specify that every first-order partial differential equation (14) has a solution depending on an arbitrary function. This solution is called general integral of the equation. We regard A as an arbitrary function of the remaining constant: $S_{GI} = S(t, q_1, ..., q_n; \alpha_1, ..., \alpha_n) + A(\alpha_1, ..., \alpha_n)$. If we impose that the $\partial S_{GI}/\partial \alpha_i = 0$ and replace the α_i by functions of co-ordinates, we obtain the general integral in terms of the arbitrary function $A(\alpha_1, ..., \alpha_n)$. When the function S_{GI} is obtained in this manner, we have

$$\frac{\partial S_{GI}}{\partial q_i} = \left.\frac{\partial S_{GI}}{\partial q_i}\right|_\alpha + \sum_k \left.\frac{\partial S_{GI}}{\partial \alpha_k}\right|_q \frac{\partial \alpha_k}{\partial q_i} = \left.\frac{\partial S_{GI}}{\partial q_i}\right|_\alpha \tag{19}$$

S_{GI} satisfies the Hamilton-Jacobi equation, since the function $S(t, q; \alpha)$ is assumed to be a complete integral of that equation.

V. FREE PARTICLE IN THE MINKOWSKI SPACETIME

Finally, we illustrate our discussion with one example, the relativistic massive free particle model. It is described by the Lagrangian

$$L = -m\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{20}$$

where x^μ is the vector in Minkowski spacetime with metric $\eta_{\mu\nu} = (-1, 1, 1, 1)$, m is the mass of the particle and τ is the parameter of the curve.

First of all, we are going to demonstrate that it is a singular system. If we calculate $(\partial L/\partial \dot{x}^\mu)$, we obtain,

$$\frac{\partial L}{\partial \dot{x}^\mu} = m\frac{\eta_{\mu\nu}\dot{x}^\nu}{\sqrt{-\eta_{\rho\xi}\dot{x}^\rho\dot{x}^\xi}} \tag{21}$$

Now, it is easy to demonstrate that the Hessian matrix is,

$$\frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = m\frac{\eta_{\mu\alpha}}{\sqrt{(-\eta_{\rho\xi}\dot{x}^\rho x^\xi)^3}}[-\delta^\alpha_\nu \dot{x}^\sigma \dot{x}_\sigma + \dot{x}^\alpha \eta_{\nu s}\dot{x}^s] \tag{22}$$

and defining $\Pi^\alpha_\nu \equiv -\delta^\alpha_\nu \dot{x}^\sigma \dot{x}_\sigma + \dot{x}^\alpha \eta_{\nu s}\dot{x}^s$, we obtain,

$$W_{\mu\nu} = \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = \frac{m}{\sqrt{(-\eta_{\rho\xi}\dot{x}^\rho \dot{x}^\xi)^3}}\eta_{\mu\alpha}\Pi^\alpha_\nu \tag{23}$$

If we calculate the determinant of the Hessian matrix (22), we obtain that it is equal to 0. Therefore, we have a singular system.

To realise the kind of problem that we know in gauge theories with the canonical formalism, we take the Lagrange equation,

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0 \tag{24}$$

and if we calculate the eq.(24), we arrive at the following expression,

$$m\frac{\eta_{\rho\mu}}{\sqrt{(-\eta_{s\gamma}\dot{x}^s\dot{x}^\gamma)^3}}[-\delta^\mu_\sigma \dot{x}^\alpha \dot{x}_\alpha + \dot{x}^\mu \eta_{\sigma\nu}\dot{x}^\nu]\ddot{x}^\sigma = 0 \tag{25}$$

As we can see, we obtain the Hessian matrix as we expected since the Lagrangian does not depend on x. Therefore, we can rewrite the eq.(25) of the following way,

$$\Pi^\mu_\sigma \ddot{x}^\sigma = 0 \tag{26}$$

As we can see, it is impossible to isolate $\ddot{x}^\sigma$ as a function of $\dot{x}^\sigma$ to obtain the equation of motion because the determinant of the Hessian matrix is equal to 0.

Notice that Π^μ_σ is a projector and it has a single null vector:

$$\left(\delta^\mu_\sigma - \frac{\dot{x}^\mu \dot{x}_\sigma}{\dot{x}^\alpha \dot{x}_\alpha}\right)\dot{x}^\sigma = 0 \tag{27}$$

As we observe, the eq.(27) is identically zero, therefore , the solution of eq.(26) is

$$\ddot{x}^\sigma = \lambda(\tau)\dot{x}^\sigma \tag{28}$$

where $\lambda(\tau)$ is an arbitrary function. This solution represents a straight line in Minkowski spacetime.

Let's see it more geometrically. As we can see, Π^μ_σ is associated with perpendicular projection of $\dot{x}^\sigma$. As a consequence of eq.(26), we realise that the projection onto $\ddot{x}^\sigma$ is equal to 0, therefore, $\ddot{x}^\sigma$ and $\dot{x}^\sigma$ has to be parallel.

In addition, we have infinite solutions due to the arbitrary function $\lambda(\tau)$. We have found a symmetry which allows us to go from one solution to other. We have solutions mathematically different but with the same physical interpretation. In other words, we can parameterize the trajectory as we want but we always have the same trajectory. This is an example of a gauge symmetry, the reparametrization invariance.

Now, we are going to observe this symmetry through Noether theory. We take $\tau' = \tau - \epsilon(\tau)$ where $\epsilon(\tau)$ is an infinitesimal arbitrary function. Making an expansion to first order, we obtain, $x^\mu(\tau) = x^\mu_N(\tau - \epsilon(\tau)) \approx x^\mu_N(\tau) - \epsilon(\tau)\dot{x}^\mu_N$. Therefore,

$$\delta x^\mu(\tau) = x^\mu_N(\tau) - x^\mu = \epsilon(\tau)\dot{x}^\mu(\tau) \tag{29}$$

For the Lagrangian variation, we derive that

$$\delta L = -\frac{\dot{x}_\mu \delta\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}} \tag{30}$$

If we derivate $\delta x^\mu(\tau)$ and we put it into eq.(30), we can obtain,

$$\delta L = \frac{d}{d\tau}\left(\epsilon(\tau)L\right) \equiv \frac{dF}{d\tau} \tag{31}$$

Therefore, $F = e(\tau)L$. The eq.(31) show us the reparametrization invariance for the Lagrangian.

To construct the Dirac's Hamiltonian, as we saw before, we need the canonical Hamiltonian and the constraint. To obtain the constraint, we take the eq.(21). It is obvious that this equation corresponds to the canonical momentum, therefore, if we calculate its norm, $p^\mu p_\mu$, we obtain,

$$p^\mu p_\mu + m^2 = 0 \tag{32}$$

In other words, the image of FL contained in T^*Q is not all T^*Q but only those q and p that obey $p^2 + m^2 = 0$. In T^*Q exists a function $\phi(q, x) = p^2 + m^2$ such that its pull-back satisfies $\phi(q, x) = 0$. We have found that $\phi(q, x)$ is a primary constraint and moreover, first class since it is unique.

Knowing that, we calculate the canonical Hamiltonian ($H_c = p^\mu \dot{x}_\mu - L$) that is,

$$H_c = 0 \tag{33}$$

So, the Dirac's Hamiltonian is,

$$H_D = \frac{1}{2}e(\tau)(p^\mu p_\mu + m^2) \tag{34}$$

where $e(\tau)$ is what we have called v^μ before and we have put a 1/2 for convenience. It easy to prove that this Hamilton has the same solution already found before. If we take the Hamilton equation, we see that p_μ is a constant and moreover, we obtain $\ddot{x}^\mu = \beta(\tau)\dot{x}^\mu$ where $\beta(\tau)$ is $(\dot{e}(\tau)m)/(\sqrt{-\dot{x}^\sigma \dot{x}_\sigma})$. This is a straight line.

We can obtain another Lagrangian associated with the Dirac's Hamiltonian as follows: first, we promote the arbitrary function e(τ) to the status of configuration variable, and second, we take its canonical momentum as a primary constraint. With these considerations, we have $H_D = \frac{1}{2}e(\tau)(p^\mu p_\mu + m^2) + \varsigma(\tau)p_e$ where $\varsigma(\tau)$ is another arbitrary function, so the Lagrangian is,

$$L(x, \dot{x}, e, \dot{e}) = p_e\dot{e} + p_\mu \dot{x}^\mu - H_D(x, p, e, p_e) \tag{35}$$

Calculating the equation of motion of $e(\tau)$ through the Dirac's Hamiltonian, we obtain that $\dot{e}(\tau) = \partial H_D/\partial p_e = \varsigma(\tau)$. If we put it into the eq.(35), finally we obtain

$$L(x, \dot{x}, e, \dot{e}) = \frac{1}{2}\frac{\dot{x}_\mu \dot{x}^\mu}{e(\tau)} - \frac{1}{2}e(\tau)m^2 \tag{36}$$

Now, let's see that this Lagrangian contains exactly the same information than eq.(20).We obtain from eq.(36),

$$\ddot{x}^\mu = \frac{d}{d\tau}\left(\ln e(\tau)\right)\dot{x}^\mu \tag{37}$$

$$e(\tau) = \frac{1}{m}\sqrt{-\dot{x}^\sigma \dot{x}_\sigma} \tag{38}$$

First of all, in the eq.(37), there is the relation between arbitrary functions, $\bar{e}(\tau) = d/d\tau\left(\ln e(\tau)\right)$. We can always change from one arbitrary function to other through this relation. In other words, the function $e(\tau)$ is a way to control the change of parametrization in the Minkowski spacetime. Moreover, introducing the eq.(38) into the eq.(37), it is easy to find the same solution that the eq.(25). Secondly, if we put the eq.(37) and eq.(38) into the eq.(36), we obtain the same Lagrangian than before. Therefore, we have not lost information during the transformation, quite the opposite, we introduce new information that we did not have before. Now, we can make the mass limit to 0. If we make that, we obtain,

$$L = \frac{1}{2}\frac{\dot{x}^\sigma \dot{x}_\sigma}{e(\tau)} \tag{39}$$

The EOM for x is the same, we have a straight line, but in the EOM for $e(\tau)$, we have

$$\dot{x}^\sigma \dot{x}_\sigma = 0 \tag{40}$$

which means that the trajectory is on the light cone. We have obtained the Lagrangian for a massless particle.

Apart from those considerations, we note that we have another primary constraint, $(\partial L/\partial e) = p_e = 0$, as we assumed before. If we calculate the Dirac's Hamiltonian, $\bar{H}_D = H_c + \lambda(\tau)p_e$, where $H_c = e(\tau)\phi$ and $\lambda(\tau)$ is another arbitrary function, these two functions are related by $\dot{e}(\tau) = \{e, H_D\} = \lambda(\tau)$. Therefore, they are not independent. ϕ is now a secondary constraint.

Now, we can imagine our starting point that both constraints are primary. Then, we construct what it is called the Extended Hamiltonian, $H_E = H_c + \sum_i e(\tau)_i\phi_i$. In our case, $H_E = e(\tau)\phi + \lambda(\tau)p_e + \mu(\tau)\phi$, and regrouping the two terms that contain ϕ, we can rewrite it of the following way,

$$H_E = \lambda(\tau)p_e + \xi(\tau)\phi = \lambda(\tau)p_e + \xi(\tau)(p_\mu p^\mu + m^2) \tag{41}$$

where and $\xi(\tau)$ and $\lambda(\tau)$ are arbitrary functions that, in this case, they are totally independent. If we calculate the Lagrangian, we obtain the same in both cases ($L = -m\sqrt{-\dot{x}^2}$) and therefore, the same Hamilton-Jacobi. We note, therefore, that Hamilton-Jacobi formalism can not distinguish between the extended Hamiltonian and the Dirac Hamiltonian.

Let us look for the solution of Hamilton-Jacobi in this case. Using the eq.(14), the eq.(34) and replacing $p_\mu = (\partial S/\partial x^\mu)$, we obtain the following expression,

$$\frac{\partial S}{\partial \tau} + \frac{1}{2}e(\tau)\left(\frac{\partial S}{\partial x_\mu}\frac{\partial S}{\partial x^\mu} + m^2\right) = 0 \tag{42}$$

As we know, this expression (the action, S) is for all $e(\tau)$, so, we can form another equation with another arbitrary function. If we subtract these two equation with different arbitrary functions, we obtain that,

$$\frac{\partial S}{\partial x_\mu}\frac{\partial S}{\partial x^\mu} = -m^2 \qquad (43)$$

and as a consequence of that,

$$\frac{\partial S}{\partial \tau} = 0 \qquad (44)$$

These equations (eq.(43) and eq.(44)) are known as the Hamilton-Jacobi equation for the free particle in Minkowski spacetime. Moreover, we note that we have two equations ((43) and (44)) instead of the usual one Hamilton-Jacobi equation for regular system. This implies that we have a free parameter less, as we mentioned before (eq.(18)).

Now, we are going to calculate the action, S. To make this, we use the eq.(10). Replacing the Lagrangian for the eq.(36) and evaluating this Lagrangian with one of its solution (eq.(38)), we find,

$$S(\tau, x; \tau_0, x_0) = -m\int_0^\tau \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau'}\frac{dx^\nu}{d\tau'}}d\tau' \qquad (45)$$

As we see, we obtain the old Lagrangian. As a consequence of this, we note that both Lagrangian (eq.(36) and eq.(20)) has the same action and therefore, the same Hamilton-Jacobi equations. Repeating the same, but now, evaluating the eq.(45) for the solution of x (a straight line), $x^\mu = x_0^\mu + v_o^\mu C(\tau)$, where $C(\tau)$ is an arbitrary function with $dC/d\tau > 0$ expressing the freedom to reparametrize the trajectory, it easy to see that, with the initial condition $x^\mu(0) = x_0^\mu$ and $x^\mu(\tau_1) = x^\mu$, we obtain,

$$S(\tau, x; x_0) = -m\sqrt{-\eta_{\mu\nu}(x^\mu - x_0^\mu)(x^\nu - x_0^\nu)} \qquad (46)$$

We have just demonstrated that the action does not depend of any evolutionary parameter (τ, τ_0) and therefore, the physical information is in the trajectory and not in the parametrization of the curve that we use.

As we saw before, if we use the constant that appears in breaking the integral action in two parts, we can always set $x_0^0 = 0$. Therefore, we write the eq.(46) (ignoring the additive constant) of the following way,

$$S(\tau, x; x_1) = -m\sqrt{(x^0)^2 - (\vec{x} - \vec{\alpha})^2} \qquad (47)$$

where $\vec{\alpha}$ is $\vec{x}_0$. Let's see the complete integral of this action, $S_{GI} = S + A(\vec{\alpha})$. We are going to study the simplest case, $A = \vec{p}\vec{\alpha}$. Consequently, we have

$$\frac{\partial S_{GI}}{\partial \vec{\alpha}} = \frac{\partial S}{\partial \vec{\alpha}} + \vec{p} = 0 \qquad (48)$$

and so, the relation $\vec{\alpha}(x^\mu, \vec{p})$ is

$$(\vec{x} - \vec{\alpha})^2 = \frac{(\vec{p})^2(x^0)^2}{m^2 + (\vec{p})^2} \qquad (49)$$

Adding the eq.(49) into the eq.(47), we derive that,

$$S_{GI}(x^\mu, \vec{p}) = -x^0\sqrt{(\vec{p})^2 + m^2} + \vec{p}\vec{x} \qquad (50)$$

Defining $p^0 = \sqrt{\vec{p}^2 + m^2}$, we obtain,

$$S_{GI}(x, p) = p_\mu x^\mu \qquad (51)$$

It is easy to demonstrate that the eq.(51) satisfies the eq.(43) and the eq.(44), and its solution that presents is exactly the same, a straight line. The Hamilton-Jacobi equation presents a lot of varieties in its form but it contains always the same information.

VI. CONCLUSIONS

We have studied some aspects of Hamilton-Jacobi theory for constrained systems (gauge theories), in the case where the constraints are first class. In general, for systems that have only first class constraints, the number of parameters in the Hamilton-Jacobi function (except the additive) is equal to the number of configuration variables minus the number of constraints.

In addition, as we have just seen in the free particle in the Minkowski spacetime, we have problems with the canonical formalism when the system is singular, therefore, we need to use an alternative way to construct this formalism. This alternative way is using the Dirac's Hamiltonian, or also called the total Hamiltonian.

Moreover, we have seen that the action, S, does not depend on any evolutionary parameter that we use to parametrize the curve. As we said before, the information is in the trajectory, not in the evolutionary parameter. The system presents a symmetry: reparametrization invariance.

Finally, using the complete integral to calculate the action, we have just seen that it presents different forms, but we have always the same information of the system.

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