Hamilton-Jacobi formalism for non-involutive systems

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Abstract

The aim of this work is to show our vision about the inconsistency of the integrability condition for non-involutive systems in the Hamilton-Jacobi formalism. In order to solve this issue, a dimensional reduction is done, showing that the reduced system contains exactly the same dynamics as the original one. Moreover, so as to give consistency to our work, we are going to display this method in four examples.

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Contents

1 Introduction

Constrained systems are one of the cornerstones of theoretical physics because every theory with gauge symmetries contains them. These constraints arise when the determinant of the Hessian matrix, which is built from the Lagrangian function, vanishes. This fact implies that there is not an inverse transformation that takes us from velocities to momenta. Two of the most important people who treated this issue were Bergmann [1] [2], who introduced the concepts of primary and secondary class constraints, and Dirac [3] who pointed out the different sorts of constraints, first and second-class (or involutive and non-involutive systems). Moreover, Dirac also introduced his canonical approach in which the constraints are added to the canonical Hamiltonian through Lagrange multipliers, and the preservation of them over the time evolution. The process of seeking all the possible constraints of the system is known as Dirac's stabilization algorithm [3].

The way of formulating the Hamilton-Jacobi formalism (HJ) in the presence of second-class constraints has been controversial. For this kind of system, the Frobenius' integrability condition is not satisfied and, consequently, the Hamilton-Jacobi partial differential equations (HJPDEs) become non-integrable. Even though there is a consensus in the formulation for involutive systems, for the case of non-involutive, some authors have elaborated alternative methods such as [4][5][6][7][8][9] in order to solve this problem.

The aim of this work is to give an interpretation of the integrability problem for the HJ formalism. We present a way out for this problem which works at least for a class of theories in which the second-class constraints have some specific features. Our proposal so as to solve this integrability issue for this particular case consists of two stages. First of all, we show that when the second-class constraints have a particular structure, we are able to get a system without them but with the same dynamics via a dimensional reduction. As a result of this procedure, the assumption of considering the HJ equations for second-class constraints as identities becomes untenable. Furthermore, for the specific kinds of second-class constraints that we are going to deal with, we show that these constraints do not play any role in the dynamics. Contrary to the role played by the first-class constraints, it is not correct to interpret them as partial differential equations (PDEs) for the Hamilton functions. Secondly and as a matter of this fact, the role of our second-class constraints is to place restrictions on the initial conditions of the dynamical trajectories. These conditions vanish in our initial formulation of the dynamics. Nevertheless, within our dimensional reduction procedure, they are allowed to take any non-vanishing constant value without affecting the solutions of the reduced dynamical system. Due to this fact, we introduce the second-class constraints as restrictions on the initial conditions for the trajectory.

Let us emphasise that the HJ theory, with the HJ equations and the Hamilton function (either complete, general...) as their solutions, is a very elaborated way to describe the solutions of a dynamical system initially formulated either in tangent space (Lagrangian formalism) or in phase space (Hamiltonian or canonical formalism).

This paper is composed by the following sections:

First of all, sec.(2) presents the HJ formalism for constrained systems. Secondly, in sec. (3) the integrability problem for non-involutive systems is shown. Then, in sec. (4) we show our vision of how the integrability problem can be solved. Afterwards, in sec.(5) we illustrate some examples where our procedure can be applied. Finally, in sec.(6) we discuss the results obtained.

2 Hamilton-Jacobi formalism

In this section, we are going to discuss the HJ formalism[1] for systems with constraints [11] [12]. Let us start considering a system in which at the times t_0 and t_1 its position is defined by the values of the co-ordinates of the configuration manifold ($\mathcal{Q}_N$), $x(t_0) = x_0$ and $x(t_1) = x_1$ respectively. The condition is such that the system moves between these two positions in such a way that the integral

$$S[x^i(t)] = \int_{t_0}^{t_1} L(t', x^i, \dot{x}^i)dt' \tag{1}$$

takes the minimum possible value keeping x_0 and x_1 fixed. This is the well-known Action Principle in the formulation of Mechanics. Once the solutions are known (or at least assuming they are known), plugging them into (1) leads to the definition of the Hamilton function $S(t_0, x_0^i, t_1, x_1^i)$.

Following Landau [10], we are going to obtain all the HJ formalism. Firstly, let us consider a small variation in the action

$$\delta S = \sum_i \left[\frac{\partial L}{\partial \dot{x}^i}\delta x^i\right]_{t_0}^{t_1} + \sum_i \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta x dt. \tag{2}$$

Since the paths satisfy the Lagrange's equations, the integral of the last part of δS is zero. Moreover, we put $\delta x(t_0) = 0$ and denote the value of $\delta x(t_1)$ by δx. Substituting $\partial L/\partial \dot{x}^i$ by p_i, we finally obtain that

$$\delta S = \sum_i p_i \delta x_i. \tag{3}$$

From the last relation, it follows that the partial derivatives of the action with respect to the co-ordinates are equal to the corresponding momenta

$$\frac{\partial S}{\partial x^i} = p_i. \tag{4}$$

Secondly, we are going to show that $\partial S/\partial t = -H_c$ where H_c is the canonical Hamiltonian. From the definition (1), its total time derivative along the path is $dS/dt = L$. Regarding S as a function of coordinates and time, and using eq.(4), we get that

$$\begin{aligned} \frac{dS}{dt} &= \frac{\partial S}{\partial t} + \sum\nolimits_i \frac{\partial S}{\partial x_i}\dot{x}_i \\ &= \frac{\partial S}{\partial t} + \sum\nolimits_i p_i \dot{x}_i \end{aligned} \tag{5}$$

and therefore, $\partial S/\partial t = L - \sum p_i \dot{x}_i$ or

$$\frac{\partial S}{\partial t} = -H_c(x_i, p_i, t). \tag{6}$$

Finally, taking the relations (6) and (4), expanding it to n variables and considering only autonomous systems, we obtain what is known as the HJ equation

$$\frac{\partial S}{\partial t} + H_c\left(x_1, ..., x_n; \frac{\partial S}{\partial x_1}, ..., \frac{\partial S}{\partial x_n}\right) = 0. \tag{7}$$

[1] For more details, one can consult [10].

The last equation (7) has the following solution,

$$S = S(t, x_1, ..., x_n; \alpha_1, ..., \alpha_n) + A \tag{8}$$

where $\alpha_1, ..., \alpha_n$ (or $x_0^1, ..., x_0^n$) and A are arbitrary constants. It seems we could have n arbitrary constants (the α's), the initial time (t_0) and one additive constant (A). As a result, one could think that we have $n + 2$ constants. However, this point of view is incorrect. We can always split the trajectory into two parts, $S(t, x; t_0, x_0) = S_2(t, x; t_1, x_1)+S_1(t_1, x_1; t_0, x_0)$ where S_1 can be considered as additive constant because t_1, x_1, t_0, x_0 are fixed parameters in our trajectory. Due to this fact, this S_1 is used so as to move the initial time t_0 to 0. Therefore, we really have $n + 1$ independent parameters, one of them additive.

Now, we take $S(t, q; \alpha)$ as the generating function and α_i as the new parameters for the complete solution. We can use S to make a canonical transformation so that α_i become the new co-ordinates and $\beta_i = -(\partial S/\partial \alpha_i)$ become the new momenta. Hence, the new Hamiltonian becomes

$$H' \equiv H_c + \frac{\partial S}{\partial t} \tag{9}$$

which vanishes because of (6). Henceforth, if we take the Hamilton's equations for H', the equations of motion are straightforward solved

$$\begin{aligned} \dot{\alpha} = 0 \quad &\rightarrow \quad \alpha = constant \\ \dot{\beta} = 0 \quad &\rightarrow \quad \beta = constant. \end{aligned} \tag{10}$$

Let us consider the case in which the Lagrangian $L(x_i, \dot{x}_i, t)$ becomes singular, i.e the determinant of the Hessian matrix W_{ij} is

$$\det W_{ij} = \det \left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j} \right) = 0. \tag{11}$$

Due to this fact, the accelerations of the variables x_i cannot be isolated as a function of their velocities and positions ($\ddot{x}_i \neq f_i(x, \dot{x})$) and therefore, the theorem about the existence and uniqueness of solutions of ordinary differential equation is no satisfied. The singularity of the Hessian matrix (W) implies the existence of primary constraints. Likewise, secondary[2] constraints can arise from the condition that the primary constraints have to be preserved in time [3]. As a result of the existence of these constraints, the dynamics is going to be restricted to a surface of lower dimensionality defined by the equations of constraints. This surface is called: the constraints' surface.

Following Dirac [3], the dynamics in the phase space can be formulated. For that, we must have a first-class Hamiltonian and a set of constraints that might be classified into first-class and second-class. First-class constraints are defined such that

$$\{\phi_\mu^{Fc}, \phi_{Any}\} \approx 0 \tag{12}$$

where $\approx$ means "weakly zero" (i.e it vanishes on the constraints' surface), ϕ_{Any} refers to any possible constraint and the Poisson bracket is defined by

$$\{A, B\} = \sum_{i=0}^{m} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} \right) \tag{13}$$

where p_i are the conjugate momenta of q_i. Moreover, second-class constraints come in an even number and are defined such that

$$\det \left| \{\phi_\mu^{Sc}, \phi_\nu^{Sc}\} \right| \neq 0. \tag{14}$$

[2] Tertiary, quaternary constraints can also appear by the stability of the previous constraints.

Notice that the concept of a function being first-class is not restricted to functions representing constraints. In fact, we say that a function f is first-class with respect to a given set of constraints if its Poisson bracket with these constraints vanishes on the constraints' surface. Geometrically this means that the vector field $X_f = \{f, -\}$ is tangent to the constraints.

Adjoining all these sorts of constraints to the HJ equation (7) (following the convencional rationale [13]), we are led to consider the coupled set of differential equations

$$\begin{aligned} &\frac{\partial S}{\partial t} + H_c\left(x_i, \frac{\partial S}{\partial x_i}\right) = 0 \\ &\phi_\mu\left(x_i, \frac{\partial S}{\partial x_i}\right) = 0 \end{aligned} \tag{15}$$

where μ goes from 1 to n and n is the number of constraints (primary, secondary, tertiary, quaternary...) that our system has. This coupled set of differential equations is called the HJPDEs.

As we have just seen, first and second-class constraints have been introduced into the HJ equation (7) as people usually do in literature [13] [14]. However, it would be more accurate for this work to introduce them inspiring us in the Landau's approach [10]. For that, let us do an analogy between the regular Landau's formalism (7) and the formalism for the case of constrained systems. As we already know [12] [3], the dynamics generator for constrained systems is the Dirac Hamiltonian,

$$H_{_D}(x,p,\lambda) = H_c(x,p) + \lambda^1_\mu \phi^\mu_{_{PFc}}(x,p) + \lambda^2_\nu \phi^\nu_{_{PSc}}(x,p) \tag{16}$$

where λ^1_μ and λ^2_ν are the Lagrange multipliers and moreover, μ and ν goes from 1 to all possible primary first-class (PFc) or primary second-class (PSc) constraints. By definition of second-class constraints, λ^2_ν become fixed by the Dirac's algorithm [3]. Because of this fact, we can define a first-class Hamiltonian such as second-class constraints are added to the canonical Hamiltonian as additional information of the dynamics, i.e

$$H_{Fc} \equiv H_c + \lambda^2_\nu \phi^\nu_{_{PSc}}. \tag{17}$$

Changing the canonical Hamiltonian H_c by the Dirac Hamiltonian $H_{_D}$ (where $H_{_D}$ contains all the dynamics for the singular system), in the eq.(7), we obtain that

$$\frac{\partial S}{\partial t} + H_{_D}\left(x_1, ..., x_n; \frac{\partial S}{\partial x_1}, ..., \frac{\partial S}{\partial x_n}\right) = 0 \tag{18}$$

and therefore,

$$\frac{\partial S}{\partial t} + H_{_{Fc}}\left(x_1, ..., x_n; \frac{\partial S}{\partial x_1}, ..., \frac{\partial S}{\partial x_n}\right) + \lambda^1_\mu \phi^\mu_{_{PFc}}\left(x_1, ..., x_n; \frac{\partial S}{\partial x_1}, ..., \frac{\partial S}{\partial x_n}\right) = 0. \tag{19}$$

Because the eq.(19) has to be satisfied $\forall\ \lambda^1_\mu$, the eq.(19) is split into a coupled set of differential equations

$$\begin{aligned} &\frac{\partial S}{\partial t} + H_{_{Fc}}\left(x_i, \frac{\partial S}{\partial x_i}\right) = 0 \\ &\phi^\mu_{_{Fc}}\left(x_i, \frac{\partial S}{\partial x_i}\right) = 0. \end{aligned} \tag{20}$$

Notice that, in the eq.(20), first-class (Fc) constraints were written instead of primary first-class (PFc) constraints. The reason is because once the primary first-class constraints are treated as identities in the HJ equations, all the constraints that can appear due to the Dirac's stabilization algorithm (secondary, tertiary and so on) will be also identities, and therefore, the action S will have to satisfy the (secondary, tertiary ...) constraints as well. For this reason, first-class constraints have to be considered, in general, in the HJ equations. Furthermore, comparing the two coupled set of differential equations (15 and 20), we notice that, for the case of the eq. (20), the second-class constraints do not appear in the HJPDEs. Therefore, inspired in Landau's approach, we observe that the second-class constraints do not participate at the same level as the first-class constraints do in the formulation of the HJ equations.

3 Integrability

In this section, we are going to discuss the integrability problem for the HJPDEs [15] and furthermore, what the integrability condition is to form a complete integrable system of partial differential equations (PDEs). In addition to this, it would be important to mention that, from now on, the convencional rationale of treating all constraints (including second-class) as identities in the HJ equations is followed.

According to [15], the necessary and sufficient condition that the HJPDEs have to obey is given by the Frobenius' integrability condition. It is based on the fact that these PDEs only admit a solution provided the constraints ϕ_μ are in strong involution,

$$\{\phi_\mu, \phi_\nu\}\big|_{p_i=\partial S/\partial x_i} = 0. \tag{21}$$

For the systems that we are considering (with second-class constraints), they do not respect this condition. They are called non-involutive systems or systems which contain second-class constraints.

Let us observe with more details the contradiction of the integrability condition for non-involutive system. For that, we are going to imagine that we have two set of second-class constraints $\phi_\mu(x,p) \approx 0$ and $\phi_\nu(x,p) \approx 0$ with their Poisson bracket

$$\{\phi_\mu, \phi_\nu\} \neq 0. \tag{22}$$

In the HJ formalism ($p_i = \partial S/\partial x^i$), the action S solves $\phi_\mu(x_i, \partial S/x_i) = 0$ as identities which do not depend on the values of the x_i variables. Therefore, treating the constraints strictly as identities and deriving them respect to one of the variables

$$\begin{aligned} \frac{\partial\phi_\mu}{\partial x^i} + \frac{\partial\phi_\mu}{\partial p_j}\frac{\partial S}{\partial x^i x^j} = 0 \\ \frac{\partial\phi_\nu}{\partial x^i} + \frac{\partial\phi_\nu}{\partial p_j}\frac{\partial S}{\partial x^i x^j} = 0 \end{aligned} \tag{23}$$

and furthermore, multiplying them by

$$\begin{aligned} \frac{\partial\phi_\nu}{\partial p_i}\left(\frac{\partial\phi_\mu}{\partial x^i} + \frac{\partial\phi_\mu}{\partial p_j}\frac{\partial S}{\partial x^i x^j}\right) = 0 \\ \frac{\partial\phi_\mu}{\partial p_i}\left(\frac{\partial\phi_\nu}{\partial x^i} + \frac{\partial\phi_\nu}{\partial p_j}\frac{\partial S}{\partial x^i x^j}\right) = 0 \end{aligned} \tag{24}$$

finally, we obtain, subtracting the two equation, that

$$\frac{\partial\phi_\nu}{\partial p_i}\frac{\partial\phi_\mu}{\partial x^i} - \frac{\partial\phi_\mu}{\partial p_i}\frac{\partial\phi_\nu}{\partial x^i} = \{\phi_\mu, \phi_\nu\} = 0. \tag{25}$$

Hence, under the assumption that S satisfies all constraints as in (15), it is observed that S should satisfy also (25) even for second-class constraints. However, we reach a contradiction due to (22). We have just illustrated basically the source of problems of second-class constrains in the HJ formalism.

4 Our Vision

As we have mentioned in the introduction, in this section we are going to introduce our vision so as to treat systems which contain second-class constraints and that, consequently, they do not satisfy the integrability conditions (21) in the HJ formalism. This proposal will be an alternative to BFT [6][7] construction (the embedding of the second-class system into a first-class one) and to the chain-chain method [5][4][8][9] (the elimination of half of the second-class constraints and the transformation of the remaining half into first-class ones). The main difference between their proposal and ours is our point of view about the step (23). They assume that the second-class constraints can be considered as identities and as a result of this consideration, they find such contradiction (25). In order to solve it, the authors seek alternative ways to overcome such drawback. Nevertheless, our approach will be different, we will show that, for a particular case of second-class constraints in the HJ formalism, half of the second-class constraints will not be able to be treated as PDEs and therefore, the contradiction (25) will not appear.

4.1 Our proposal

In order to tackle the integrability problem for second-class constraints, we address systems for which the second-class constraints are grouped in canonical pairs with the following structure,

$$\begin{aligned} \Pi_i &\equiv \pi_{y_i} \approx 0 \\ Y_i &\equiv y_i - f_i(x, \pi_x) \approx 0 \end{aligned} \tag{26}$$

where π_{y_i} are the canonical conjugates of y_i and i runes from 1 to half the number of second-class constraints. Likewise, it is important to mention that the procedure of seeking all the constraints of the system is carried out by the Dirac's stabilisation algorithm [3]. In addition to this, we should clarify that a canonical transformation can be made in order to bring the theory to this setting (26), as we will see in the example 3. Notice that it would be enough to find the canonical transformation in which $\pi_{y_i} = 0$, since this condition would directly imply that the remaining second-class constraints can be written as the second equation (Y_i).

Before introducing our lema, let us clarify our dimensional reduction. As we have commented in the introduction, such reduction will be used to show that the dynamics of the system is not affected by second-class constraints. First of all, we are going to start with a function g which depends on (x, π_x, y, π_y) and lives in the Phase space. Imposing the specific structure of second-class constraints (26) into this function (this process was called cs), the function g becomes $\bar{g}$ which only depends on (x, π_x) and lives on the second-class constraints' surface.

Namely,

$$g(x, \pi_x, y, \pi_y) \longrightarrow \bar{g}(x, \pi_x) = g(x, \pi_x, f(x, \pi_x), 0) \equiv g(x, \pi_x, y, \pi_y)\big|_{cs} \tag{27}$$

Lemma 4.1. *Being H_{Fc} a first-class Hamiltonian and $\bar{H}$ the Hamiltonian in which the constraints have been strongly implemented (26), we can assert that these Hamiltonians contain the same Hamilton's equations if the constrains have this specific structure (26) :*

$$\left.\frac{\partial H_{Fc}}{\partial x_i}\right|_{sc} \approx \frac{\partial \bar{H}_{Fc}}{\partial x_i} \qquad\qquad \left.\frac{\partial H_{Fc}}{\partial \pi_i}\right|_{sc} \approx \frac{\partial \bar{H}_{Fc}}{\partial \pi_i} \tag{28}$$

In order to prove this lemma, let us show that the Hamilton's equations contain the same dynamics. By definition (27), we know that

$$\bar{\phi}_{Fc}(x,\pi_x) = \phi_{Fc}(x,\pi_x, y = f(x,\pi_x), 0) \equiv \phi_{Fc}(x,\pi_x,y,\pi_y)|_{cs} \tag{29}$$

and therefore, the Hamilton's equations read

$$\begin{aligned} \dot{x} = \{x, \bar{H}_{Fc}\} = \frac{\partial \bar{H}_{Fc}}{\partial \pi_x} &= \left.\frac{\partial H_{Fc}}{\partial \pi_x}\right|_{cs} + \left.\frac{\partial H_{Fc}}{\partial y}\right|_{cs} \left.\frac{\partial f}{\partial \pi_x}\right|_{cs} \\ \dot{\pi}_x = \{\pi_x, \bar{H}_{Fc}\} = -\frac{\partial \bar{H}_{Fc}}{\partial x} &= -\left(\left.\frac{\partial H_{Fc}}{\partial x}\right|_{cs} + \left.\frac{\partial H_{Fc}}{\partial y}\right|_{cs} \left.\frac{\partial f}{\partial x}\right|_{cs} \right). \end{aligned} \tag{30}$$

On the other hand, as we know that $\pi_y \approx 0$, its Poisson bracket becomes

$$\{\pi_y, H_{Fc}\}|_{cs} = \frac{\partial H_{Fc}}{\partial y} \approx 0 \tag{31}$$

and, as a consequence, the derivative (31) has to be proportional to a first-class constraint [3],

$$\frac{\partial H_{Fc}}{\partial y} \propto \phi^{Fc}. \tag{32}$$

As a result of this, the equations of motion are

$$\begin{aligned} \frac{\partial \bar{H}_{Fc}}{\partial \pi} &= \left.\frac{\partial H_{Fc}}{\partial \pi_x}\right|_{cs} + \phi^{Fc} \left.\frac{\partial f}{\partial \pi_x}\right|_{cs} \\ \frac{\partial \bar{H}_{Fc}}{\partial x} &= \left.\frac{\partial H_{Fc}}{\partial x}\right|_{cs} + \phi^{Fc} \left.\frac{\partial f}{\partial x}\right|_{cs}. \end{aligned} \tag{33}$$

Finally, since we know that $\phi^{Fc} \approx 0$, the equations (33) read

$$\begin{aligned} &\boxed{\frac{\partial \bar{H}_{Fc}}{\partial \pi} \approx \left.\frac{\partial H_{Fc}}{\partial \pi_x}\right|_{cs}} \\ &\boxed{\frac{\partial \bar{H}_{Fc}}{\partial x} \approx \left.\frac{\partial H_{Fc}}{\partial x}\right|_{cs}} \end{aligned} \tag{34}$$

Let us prove that the reduced constraints are first-class

$$\{\bar{\phi}^{Fc}_i, \bar{\phi}^{Fc}_j\} \approx \{\phi^{Fc}_i, \phi^{Fc}_j\}|_{cs} \approx 0. \tag{35}$$

So,

$$\{\bar{\phi}^{Fc}_i, \bar{\phi}^{Fc}_j\} = \frac{\partial \bar{\phi}^{Fc}_i}{\partial x}\frac{\partial \bar{\phi}^{Fc}_j}{\partial \pi_x} - \frac{\partial \bar{\phi}^{Fc}_i}{\partial \pi_x}\frac{\partial \bar{\phi}^{Fc}_j}{\partial x}. \tag{36}$$

Taking into account that we have a function f that depends on x and π_x, the derivatives are

$$\begin{aligned}\frac{\partial\bar{\phi}^{Fc}}{\partial x} &= \left.\frac{\partial\phi^{Fc}}{\partial x}\right|_{cs} + \left.\frac{\partial\phi^{Fc}}{\partial y}\right|_{cs}\frac{\partial f}{\partial x}\\ \frac{\partial\bar{\phi}^{Fc}}{\partial \pi_x} &= \left.\frac{\partial\phi^{Fc}}{\partial \pi_x}\right|_{cs} + \left.\frac{\partial\phi^{Fc}}{\partial y}\right|_{cs}\frac{\partial f}{\partial \pi_x}.\end{aligned} \tag{37}$$

On the other hand, since the constraint is first-class implies that

$$\begin{aligned}\left.\{\phi^{Fc}, y\}\right|_{cs} &= \left.\frac{\partial\phi^{Fc}}{\partial y}\right|_{cs} \approx 0\\ \left.\{\phi^{Fc}, \pi_y\}\right|_{cs} &= \left.\frac{\partial\phi^{Fc}}{\partial \pi_y}\right|_{cs} \approx 0\end{aligned} \tag{38}$$

and, for the same reason as before, this derivatives become

$$\begin{aligned}\left.\frac{\partial\phi^{Fc}}{\partial y}\right|_{cs} &\propto \phi^{Fc}\\ \left.\frac{\partial\phi^{Fc}}{\partial \pi_y}\right|_{cs} &\propto \phi^{Fc}.\end{aligned} \tag{39}$$

Thus,

$$\begin{aligned}\{\bar{\phi}_i^{Fc}, \bar{\phi}_j^{Fc}\} &= \left(\left.\frac{\partial\phi_i^{Fc}}{\partial x}\right|_{cs} + \left.\phi_i^{Fc}\right|_{cs}\frac{\partial f}{\partial x}\right)\left(\left.\frac{\partial\phi_j^{Fc}}{\partial \pi_x}\right|_{cs} + \left.\phi_j^{Fc}\right|_{cs}\frac{\partial f}{\partial \pi_x}\right)\\ &- \left(\left.\frac{\partial\phi_i^{Fc}}{\partial \pi_x}\right|_{cs} + \left.\phi_i^{Fc}\right|_{cs}\frac{\partial f}{\partial \pi_x}\right)\left(\left.\frac{\partial\phi_j^{Fc}}{\partial x}\right|_{cs} + \left.\phi_j^{Fc}\right|_{cs}\frac{\partial f}{\partial x}\right)\\ &= \left.\frac{\partial\phi_i^{Fc}}{\partial x}\right|_{cs}\left.\frac{\partial\phi_j^{Fc}}{\partial \pi_x}\right|_{cs} - \left.\frac{\partial\phi_i^{Fc}}{\partial \pi_x}\right|_{cs}\left.\frac{\partial\phi_j^{Fc}}{\partial x}\right|_{cs} + \left.\phi_i^{Fc}\right|_{cs}\left(\left.\frac{\partial\phi_j^{Fc}}{\partial \pi_x}\right|_{cs}\frac{\partial f}{\partial x}\right.\\ &- \left.\left.\frac{\partial\phi_j^{Fc}}{\partial x}\right|_{cs}\frac{\partial f}{\partial \pi_x}\right) + \left.\phi_j^{Fc}\right|_{cs}\left(\left.\frac{\partial\phi_j^{Fc}}{\partial \pi_x}\right|_{cs}\frac{\partial f}{\partial x} - \left.\frac{\partial\phi_j^{Fc}}{\partial x}\right|_{cs}\frac{\partial f}{\partial \pi_x}\right)\\ &= \left.\{\phi_i^{Fc}, \phi_j^{Fc}\}\right|_{cs} - \phi_i^{Fc}\left.\{\phi_j^{Fc}, f\}\right|_{cs} - \phi_j^{Fc}\left.\{\phi_i^{Fc}, f\}\right|_{cs}.\end{aligned} \tag{40}$$

Finally, since we know that $\phi^{Fc} \approx 0$ and $\left.\{\phi_i^{Fc}, \phi_j^{Fc}\}\right|_{cs} \approx 0$, the equation (40) reads

$$\boxed{\{\bar{\phi}_i^{Fc}, \bar{\phi}_j^{Fc}\} \approx 0} \tag{41}$$

Let us also prove that $\bar{H}_{Fc}$ is (Fc) first-class in the reduced Phase space

$$\{\bar{\phi}^{Fc}, \bar{H}_{Fc}\} \approx \left.\{\phi^{Fc}, H_{Fc}\}\right|_{cs} \approx 0. \tag{42}$$

So, applying the relations (34) and (37), we find that

$$\begin{aligned}\{\bar{\phi}^{Fc}, \bar{H}_{Fc}\} &= \frac{\partial\bar{\phi}^{Fc}}{\partial x}\frac{\partial\bar{H}_{Fc}}{\partial \pi_x} - \frac{\partial\bar{\phi}^{Fc}}{\partial \pi_x}\frac{\partial\bar{H}_{Fc}}{\partial x}\\ &\approx \left(\left.\frac{\partial\phi^{Fc}}{\partial x}\right|_{cs} + \left.\phi^{Fc}\right|_{cs}\frac{\partial f}{\partial x}\right)\left.\frac{\partial H_{Fc}}{\partial \pi_x}\right|_{cs} - \left(\left.\frac{\partial\phi^{Fc}}{\partial \pi_x}\right|_{cs} + \left.\phi^{Fc}\right|_{cs}\frac{\partial f}{\partial \pi_x}\right)\left.\frac{\partial H_{Fc}}{\partial x}\right|_{cs}.\end{aligned} \tag{43}$$

All the terms that are multiplied by ϕ^{Fc} will vanish on the constraints' surface. Therefore, the only terms that survive are

$$\begin{aligned} \{\bar{\phi}, \bar{H}_{Fc}\} &\approx \left.\frac{\partial \phi}{\partial x}\right|_{cs} \left.\frac{\partial H_{Fc}}{\partial \pi_x}\right|_{cs} - \left.\frac{\partial \phi}{\partial \pi_x}\right|_{cs} \left.\frac{\partial H_{Fc}}{\partial x}\right|_{cs} \\ &\boxed{\{\bar{\phi}, \bar{H}_{Fc}\} \approx \{\phi^{Fc}, H_{Fc}\}\big|_{cs} \approx 0} \end{aligned} \tag{44}$$

We have just observed that, making a dimensional reduction, the dynamics of the reduced system is not affected and, as a consequence, the two Hamiltonians present the same equations of motion. In the appendix A, there is another way to show that this lemma works.

4.2 Brief comments about our proposal

As we have just shown, the H_{Fc} and $\bar{H}_{Fc}$ have the same Hamilton's equations for the reduced variables. Because of this fact, we can obtain a system without second-class constraint but with the same dynamics via a dimensional reduction. In addition to this, the integrability conditions in the H-J formalism will be satisfied because the second-class constraints will not appear in the HJPDEs.

The first observation is that, for a HJ theory without second-class constraints, the action does not have dependence on the isolated configuration variables (y_i). Thus, one could think that, through this procedure, we might not solve the dynamics for these variables. However, there is not problem in this. Once we have solved the equation of motion for the other variables, the dynamics of the isolated variables will be totally determined due to the second-class constraints. Therefore, one can observe that Y_i (26) are not identities, but equations for the canonical variables y_i. Because of this fact, in the HJ formalism, the contradiction, which we have seen in the section 3, would not occur. The step (23) would be absurd.

The second observation is that if we rewrite (26) as,

$$\begin{aligned} &\Pi_i \equiv \pi_{y_i} = 0 \\ &Y_i \equiv y_i - f_i(x, \pi_x) = constant \end{aligned} \tag{45}$$

the new set of second-class constraints would be as good as the original ones, since the proof would not change. Consequently, this leads to think that Y_i do not play any role neither in the dynamics nor in the gauge structure of the theory. The second-class constraints Y_i are simply restrictions on the initial conditions of the systems. Therefore, the reduced theory (the dynamics and the initial conditions) gives an evolution of the system in which the first-class and the second-class constraints are satisfied at any time.

The last comment that we are forced to say, as one can easily notice, is the limitation of this method. This alternative of treating the second-class constraints works it the second-class constraints have a specific structure (26), as we have mentioned above. If it is not case and we consider a system in which the structure of the constraints is not like ours, we could not assert that this method works. We would have to use an alternative such as BFT construction [6][7] or a canonical transformation that brings us to (26).

5 Examples

In order to illustrate our proposal, we are going to introduce four examples. We have to say that all of them are done by BFT construction as well [14]. Only in the first example we are going to show explicitly the integrability problem for the HJ formalism so as to realise what the problem is and how it is overcome.

5.1 Example 1

Consider the Lagrangian [14]

$$\mathcal{L} = \frac{1}{2}\dot{x}y^2 - y. \tag{46}$$

The following equations of motion of the system can be obtained through the Euler-Lagrange equations,

$$\begin{aligned} y(t) &= y_0 \\ x(t) &= \frac{1}{y_0}t + x_0 \end{aligned} \tag{47}$$

where y_0 and x_0 are the initial positions of each variable. It is important to remark that $\dot{x} = 1/y$ determines the equation of motion for $y(t)$, once the equation of motion for $x(t)$ is known. The canonical Hamiltonian is

$$H_c = \frac{1}{y}\pi_x + \frac{1}{2}y \tag{48}$$

with the constraints

$$\begin{aligned} \phi_1 &\equiv \pi_y \approx 0 \\ \phi_2 &\equiv y - \sqrt{2\pi_x} \approx 0 \end{aligned} \tag{49}$$

where ϕ_2 is obtained due to the stability of the constraint ϕ_1. We can instantly notice that its Poisson bracket, $\{\phi_1, \phi_2\} = -1 \neq 0$, does not vanish and therefore, we are facing a non-involutive system.
In the HJ formalism, the HJPDEs are

$$\frac{\partial S}{\partial t} + \frac{1}{y}\frac{\partial S}{\partial x} + \frac{1}{2}y = 0 \tag{50}$$

$$\frac{\partial S}{\partial y} = 0 \tag{51}$$

$$y - \sqrt{2\frac{\partial S}{\partial x}} = 0. \tag{52}$$

Notice that the eq's. (50, 51 and 52) are inconsistent when the eq. (52) is considered as identity in the HJ equations. It would not make sense to make the derivative with respect to y of the eq.(52), since we would obtain an untenable result (1=0). Thus, in the second-class formulation, the set of coupled Hamilton-Jacobi equations does not admit solution (they do not satisfy the integrability condition (21)) if all the constraints are treated as identities in the HJ equations. However, this is a good example to illustrate that if the eq.(52) is considered as a equation to determinate the variable y instead of an identity in the HJ equations, the inconsistency disappears. The reason is because the eq. (52) can be plugged into the eq. (50) and therefore, the eq. (51) and the eq. (50) (once the y is substituted) would be totally consistent. This is in fact an anticipation of the dimensional reduction procedure, which we will address bellow.
So, if we implement strongly the constraint ϕ_2 into the canonical Hamiltonian, we get that the Hamiltonian for the reduced set of variables is

$$\bar{H}_c = \sqrt{2\pi_x} \tag{53}$$

It is straightforward to observe that, if we compute the Hamilton's equations, we obtain the same equations of motion (47),

$$\begin{aligned} x(t) &= x_0 + \frac{1}{\sqrt{2\pi_{x_0}}}(t-t_0) \\ y(t) &= \sqrt{2\pi_{x_0}} \\ \pi_x(t) &= \pi_{x_0} \\ \pi_y(t) &= 0 \end{aligned} \tag{54}$$

where x_0, π_{x_0} and y_0 are the initial conditions of each variable at $t = t_0$.
Likewise, the HJ equation takes the form

$$\frac{\partial S}{\partial t} + \sqrt{2\frac{\partial S}{\partial x}} = 0. \tag{55}$$

The eq. (55) can be solved indirectly by considering the eq.(1). Thus, if the solutions (47) are plugged into the action (1), where the reduced Lagrangian is

$$L(\dot{x}) = -\frac{1}{2}\frac{1}{\dot{x}}, \tag{56}$$

we observe that the Hamilton function is

$$S(t, t_0, x, x_0) = -\frac{1}{2}\frac{(t-t_0)^2}{(x-x_0)}. \tag{57}$$

Using the HJ formalism, we find the same equation of motion for $x(t)$

$$\frac{\partial S}{\partial x_0} = -\beta \qquad \Rightarrow \qquad x - x_0 = \frac{1}{\sqrt{2\beta}}(t-t_0) \tag{58}$$

Notice that, for the same reason as before, all the equations of motion for all the variables can be recovered through the relations of ϕ_2, $\dot{x}(t) = 1/y(t)$ and $\partial S/\partial y = \pi_y$. It is easy to show that eq.(57) is solution of the HJ eq.(55). Furthermore, applying the derivatives with respect y in the eq.(57) and introducing by hand ϕ_2, we find that

$$\begin{aligned} &\frac{\partial S}{\partial y} = 0 \\ &\frac{\partial S}{\partial x} - \frac{1}{2}y^2 = 0 \end{aligned} \tag{59}$$

All in all, we illustrate that, making the dimensional reduction, we can obtain the same dynamics for the reduced system.

5.2 Example 2

Consider the Lagrangian [14]

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2 \tag{60}$$

which contains the following equations of motion,

$$\left.\begin{aligned} \ddot{x} + \dot{y} + x - y = 0 \\ \dot{x} + x - y = 0 \end{aligned}\right\} \qquad \begin{aligned} x(t) &= x_0 + \dot{x}_0(t-t_0) \\ y(t) &= \dot{x}_0 + x_0 + \dot{x}_0(t-t_0) \end{aligned} \tag{61}$$

where x_0, $\dot{x}_0$ are the initial values of each variable. It is important to remark that the relation $\pi_x = \dot{x} + y$ determines the dynamics of y, once the dynamics of the x and π_x are solved.
The canonical Hamiltonian is

$$H_c = \frac{1}{2}(\pi_x - y)^2 + \frac{1}{2}(x - y)^2 \tag{62}$$

with the constraints

$$\phi_1 \equiv \pi_y \approx 0 \qquad \phi_2 \equiv y - \frac{1}{2}(\pi_x + x) \approx 0 \tag{63}$$

where ϕ_2 is found due to the stability of the constraint ϕ_1. We can observe that its Poisson bracket, $\{\phi_1, \phi_2\} = -1 \neq 0$, does not vanish and therefore, this example is a non-involutive system.
As we can observe, the structure of constraints (63) already have our structure (26) to apply directly the dimensional reduction. Hence, if the constraints are strongly implemented into the Hamiltonian (62), we obtain that

$$\bar{H}_c = \frac{1}{4}(\pi_x - x)^2. \tag{64}$$

Likewise, the HJ-equation takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{4}\left(\frac{\partial S}{\partial x} - x\right)^2 = 0. \tag{65}$$

Let us obtain again the action through (1). Plugging the solutions of the equations of motion into the reduced Lagrangian

$$L(x, \dot{x}) = \dot{x} + \dot{x}x, \tag{66}$$

the Hamilton function becomes

$$S(t, t_0, x, x_0) = \frac{1}{2}(x^2 - x_0^2) + \frac{(x - x_0)^2}{t - t_0}. \tag{67}$$

Through the HJ formalism, the equation of motion for $x(t)$ is

$$\frac{\partial S}{\partial x_0} = \beta \qquad \Rightarrow \qquad x(t) = x_0 - \frac{1}{2}(\beta + x_0)(t - t_0) \tag{68}$$

which is exactly the same dynamics as we have found before. Making the derivative with respect to y and introducing by hand ϕ_2, we can also recover all the Hamilton-Jacobi equations for this system

$$\begin{aligned} &\frac{\partial S}{\partial y} = 0 \\ &\frac{\partial S}{\partial x} + x - 2y = 0. \end{aligned} \tag{69}$$

All in all, this system is totally integrable.

5.3 Example 3: Landau model in the zero mass limit

Let us consider the following Lagrangian [14]

$$L = \frac{B}{2}\vec{q} \times \dot{\vec{q}} - \frac{k}{2}\vec{q}^2 \tag{70}$$

which describes a spinless charged massless particle moving on a two dimensional plane in a constant magnetic field B perpendicular to it. Moreover, recall that (in two dimensions) $\vec{q} \times \dot{\vec{q}} = \sum_{i,j=1}^{2} \epsilon_{ij} q^i \dot{q}^j$. One can obtain (through the Euler-Lagrange equations) that the dynamics of the system is

$$\begin{aligned} q_2(t) &= A \sin(wt) + \bar{B}\cos(wt) \\ q_1(t) &= A\cos(wt) - \bar{B}\sin(wt) \end{aligned} \tag{71}$$

where $w = k/B$ and, A and $\bar{B}$ are

$$\begin{aligned} A &= -\csc\left[w\left(t_0 - t_1\right)\right]\left(q_2^1 \cos(wt_0) - q_2^0 \cos(wt_1)\right) \\ \bar{B} &= \csc\left[w\left(t_0 - t_1\right)\right]\left(q_2^1 \sin(wt_0) - q_2^0 \sin(wt_1)\right) \end{aligned} \tag{72}$$

with the initial conditions $q_2(t_0) \equiv q_2^0$ and $q_2^1(t_1) \equiv q_2^1$. Moreover, the constraints of the system are

$$\phi_i \equiv \frac{1}{\sqrt{B}} p_i + \frac{\sqrt{B}}{2}\epsilon_{ij} q^j \approx 0. \tag{73}$$

For this case, the structure of constraints (73) is not like (26) so as to apply directly the dimensional reduction. Because of this, a canonical transformation can be made in which the structure (73) is transformed into ours (26). This canonical transformation is

$$p_y = p_1 + \frac{B}{2} q_2 \qquad y = q_1 \qquad x = q_2 \qquad p_x = p_2 + \frac{B}{2} q_1. \tag{74}$$

Due to this canonical transformation, the constraint structure is modified into

$$\begin{aligned} \bar{\phi}_1 &\equiv p_y \approx 0 \\ \bar{\phi}_2 &\equiv y - \frac{1}{B} p_x \approx 0 \end{aligned} \tag{75}$$

which follows the same structure that (26). Furthermore, the canonical Hamiltonian in terms of the new coordinates becomes

$$H_c = \frac{k}{B}\left(y p_x - \frac{B}{2} y^2 - p_y x + \frac{B}{2} x^2\right). \tag{76}$$

Applying strongly the new constraints (75), we obtain that

$$\bar{H}_c = \frac{k}{2B^2}\left(p_x^2 + B^2 x^2\right) \tag{77}$$

which contains the same equation of motion

$$\begin{aligned} \dot{x} &= \{x, \bar{H}_c\} = \frac{k}{B^2} p_x \\ \dot{p}_x &= \{p_x, \bar{H}_c\} = -kx. \end{aligned} \tag{78}$$

Notice that the eq.(77) is the Hamiltonian of the harmonic oscillator. On the other hand, the HJ equation becomes

$$\frac{\partial S}{\partial t} + \frac{k}{2B^2}\left[\left(\frac{\partial S}{\partial x}\right)^2 + B^2x^2\right] = 0. \tag{79}$$

Let us compute the action for the reduced system. For that, the Lagrangian for the reduced system is

$$L(t,x,\dot{x}) = \frac{1}{2}\frac{B^2}{k}\dot{x}^2 - \frac{k}{2}x^2 \tag{80}$$

where the eq's. (78 and 77) were used. Therefore, applying the definition of the action (1) and using the equation of motion for x(t), which its dynamics is exactly the same as q_2, we obtain that the action is

$$S(x,x_0,t,t_0) = -\frac{1}{2}B\csc\left(w(t_0-t)\right)\left[(x_0^2+x^2)\cos(w(t_0-t)) - 2x_0x\right]. \tag{81}$$

Through the HJ formalism, the equation of motion for $x(t)$ is

$$\frac{\partial S}{\partial x_0} = \beta \qquad \Rightarrow \qquad x(t) = \frac{\beta}{B}\sin(w(t-t_0)) + x_0\cos(w(t-t_0)) \tag{82}$$

which is exactly the same dynamics as before. Again, notice that we can recover all the equation of motion for all the variables through the constraints.

5.4 Example 4: Multidimensional rotator

This example was done in [16] through BFT construction, however, we are going to explain it through the dimensional reduction. Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\vec{q}}^2 + \lambda\vec{q}\cdot\dot{\vec{q}} \tag{83}$$

which effectively describes the motion on a $n-1$ dimensional sphere without specification of the radius of the sphere. The canonical Hamiltonian is

$$H_c = \frac{1}{2}(\vec{\pi} - \lambda\vec{q})^2 \tag{84}$$

with the constraints

$$\phi_1 \equiv \pi_\lambda \approx 0 \qquad \phi_2 \equiv \lambda - \frac{\vec{\pi}\cdot\vec{q}}{\vec{q}^2} \approx 0 \tag{85}$$

where ϕ_1 is a primary constraint and ϕ_2 is a secondary constraint due to the stability of ϕ_1. Moreover, it easy to realise that its Poisson bracket does not vanish, $\{\phi_1,\phi_2\} = -1$, and therefore, this system is a non-involutive system. It is important to observe that the structure of constraints are already satisfied (26) and, consequently, the method can be directly used. Hence, applying strongly the constraints into the Hamiltonian (84), we obtain that

$$\bar{H}_c = \frac{1}{2}\left(\vec{\pi}^2 - \frac{(\vec{\pi}\cdot\vec{q})^2}{\vec{q}^2}\right) \tag{86}$$

and therefore, the HJ equation reads

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q_a}\right)^2 - \frac{1}{2}\left(\frac{\partial S}{\partial q_a}\right)\left(\frac{\partial S}{\partial q_b}\right)\frac{q_aq_b}{\vec{q}^2} = 0 \tag{87}$$

where $q_a q_a = \vec{q}^2$. Comparing the eq. (87) with the equation obtained in [16] [14], we realise that the result is exactly the same and therefore, our proposal is equivalent. Due to this fact, the procedure in order to find the solution for the action S will be followed.
Following [16] [14], we make the Ansatz,

$$S = \mathcal{F}(\vec{n} \cdot \vec{q}) \tag{88}$$

with $\vec{n}$ a unit normal vector parametrized by $n-1$ constants. Notice that the normalisation of $\vec{n}$ implies for the n'th component, $n_n = \sqrt{1 - \sum_{a=1}^{n-1} n_a^2}$. Since the Hamiltonian does not depend explicitly on time ($\partial S/\partial t = -\alpha^2$), the eq. (87) for $\mathcal{F}$ becomes,

$$\frac{1}{2}\left(1 - \frac{(\vec{n} \cdot \vec{q})^2}{\vec{q}^2}\right)\mathcal{F}'^2(\vec{n} \cdot \vec{q}) = \frac{\alpha^2}{2}. \tag{89}$$

Setting $x = \vec{n} \cdot \vec{q}$ and $r^2 = \vec{q}^2$, we obtain that

$$\mathcal{F}'(x) = \pm \frac{\alpha}{\sqrt{1 - \frac{x^2}{r^2}}} \tag{90}$$

and, once the eq.(90) is solved, the action takes the form

$$S = \alpha\, r\, tan^{-1}\left(\frac{\vec{n} \cdot \vec{q}}{\sqrt{r^2 - (\vec{n} \cdot \vec{q})^2}}\right) - \frac{\alpha^2}{2}t + A \tag{91}$$

where A is the integration constant. Applying the H-J formalism yields the n-independent new coordinates:

$$\frac{\partial S}{\partial \alpha} = \beta \qquad \frac{\partial S}{\partial n_a} = \beta_a. \tag{92}$$

From the first equation, we can obtain

$$\vec{n} \cdot \vec{q} = r \cos\left(\frac{\beta + \alpha t}{r}\right) \equiv r \cos \Omega(t). \tag{93}$$

Using this relation, we get from the second equation (92) that

$$q_a = \frac{\beta_a}{\alpha} \sin \Omega(t) + \frac{n_a q_n}{n_n}. \tag{94}$$

Multiplying it by n_a, summing from $a = 1$ to $a = n-1$ and using the normalisation of $\vec{n}$, we find that

$$\vec{n} \cdot \vec{q} = \frac{1}{\alpha} \sum_{a=1}^{n-1} n_a \beta_a \sin \Omega(t) + \frac{q_n}{n_n} = r \cos \Omega(t). \tag{95}$$

Therefore, q_n can be isolated through (95) :

$$q_n = n_n \left(r \cos \Omega(t) - \frac{1}{\alpha} \sum_{a=1}^{n-1} n_a \beta_a \sin \Omega(t) \right). \tag{96}$$

Finally, plugging the eq.(96) into the eq. (94), we find the following relations

$$\begin{aligned} q_a &= \frac{1}{\alpha}\left(\beta_a - n_a \sum_{a=1}^{n-1} n_a \beta_a\right) \sin\Omega(t) + n_a r \cos\Omega(t) \\ r &= \sqrt{\frac{\vec{\beta}^2 - (\vec{n}\cdot\vec{\beta})^2}{\alpha^2}} \end{aligned} \tag{97}$$

which are exactly the same solutions that [14][16] found by BFT construction.

6 Final remarks

The presence of second-class constrains leads to a problem in the Hamilton-Jacobi (HJ) theory for constrained systems. Such a problem is embodied by the violation of the integrability condition [15]. As in Example 1 is observed, the differential equations (HJPDE) are in direct conflict among themselves.

A possible method to treat these kinds of systems is the BFT [6][7] construction. This method consists of the formulation of a consistent set of HJ equations and the conversion of the second-class constraints into first class ones characterised by a strongly involutive algebra. Another method, as an alternative, would be the chain-chain method [4][5][8][9] which consists of the elimination of half of the second-class constraints transforming them into first-class ones. However, these methods are not followed in this work. A new vision is showed about the role that the second-class plays in the dynamics of the constrained system in the HJ formalism.

Having the following set of second-class constraints,

$$\begin{aligned} \pi_{y_i} &= 0 \\ y_i - f_i(x, \pi_x) &= 0, \end{aligned}$$

a dimensional reduction can be done such that

$$g(x, \pi_x, y, \pi_y) \longrightarrow \bar{g}(x, \pi_x) = g(x, \pi_x, f(x, \pi_x), 0) \equiv g(x, \pi_x, y, \pi_y)\big|_{cs}$$

where $g(x, \pi_x, y, \pi_y)$ is an arbitrary function in the Phase space and $\bar{g}(x, \pi_x)$ is an arbitrary function in which the second-class constraints are applied. Doing such reduction, we can assert that the Hamilton's equations for the variables of the reduced system (x and π_x) are exactly the same as the original system, namely

$$\left.\frac{\partial H_{Fc}}{\partial x_i}\right|_{sc} \approx \frac{\partial \bar{H}_{Fc}}{\partial x_i} \qquad\qquad \left.\frac{\partial H_{Fc}}{\partial \pi_i}\right|_{sc} \approx \frac{\partial \bar{H}_{Fc}}{\partial \pi_i}.$$

Moreover, as we mentioned before, the structure of the second-class constraints can be rewritten such as (45) without modifying the proof. As a result of this, the second-class constraints do not play role neither in the dynamics nor in the gauge structure of the theory. They only contribute to set the restrictions of the initial conditions as we had been able to suspect following the Landau's approach [10].

As we have observed in the examples, the second relevant consequence is that the problem of integrability has been solved. We have noticed that once the second-class constraints have been implemented into the Hamiltonian, they do not show up in the HJPDEs. As a result of this, the integrability condition in the HJ formalism is now satisfied.

Due to the dimensional reduction, the action does not have any dependence on the isolated canonical variables (y_i). Consequently, the constraints Π_i are already implemented and the constraints Y_i are not identities, but equations for these isolated canonical variables.

All in all, what we are trying to say is that the integrability condition problem of the second-class constraints comes from treating the second-class constraints (Y_i) as identities. As we have seen, Y_i are not identities in the HJ equations but they are equations for the isolated variables (y_i). Because of this fact, the procedure (25) cannot be applied, since the step (23) would not make sense. As a result of this, there would not be any contradiction at all. It is important to remark that we are not saying that there are not second-class constraints, but this sort of constraint is not a problem in order to satisfy the integrability condition.

Finally, we must mention that there is an important limitation in this method. As we have said before, this alternative proposal of treating the problem of integrability for systems in which constraints of second class appear only works for a specific structure of constraints (26). Thus, if we address a system which contains a different structure (26), we can not assert that this method works. It is a still open issue.

7 Future work

It would be as a future work to try to prove this method for the general case, i.e where the structure of second-class constraints is modified in such a way that π_{y_i} is not equal to zero, but $\pi_{y_i} - g_i(x, \pi_x) = 0$.

8 Appendix A

In this appendix, let us show another way to prove that the dimensional reduction works.

Dynamics for constrained systems

Before showing it, let us introduce a redefinition of our Hamiltonian so as to make the Dirac bracket unnecessary.

As we already know from the dynamics of constrained systems [12], the Hamiltonian time evolution vector field is given by

$$\mathbf{X_H} := \frac{\partial}{\partial t} + \{-, H_c\} + v^\mu\{-, \phi_\mu\}, \tag{98}$$

where v^μ are arbitrary functions of time, $\{-,-\}$ is the Poisson bracket and we introduce $\partial/\partial t$ to account possible explicit dependences on time. The requirement of the tangency of $\mathbf{X_H}$ to the second class constraints fixes some arbitrariness in the Hamiltonian dynamics. The arbitrary functions v^{ν_1}, where ν_1 runs over the indices of the secondary constraints, become determined as canonical functions $v_c^{\nu_1}$ through

$$0 = \mathbf{X_H}\phi_{\mu_1} = \{\phi_{\mu_1}, H_c\} + v_c^{\nu_1}\{\phi_{\mu_1}, \phi_{\nu_1}\}, \tag{99}$$

where yields

$$v_c^{\nu_1} = -M^{\nu_1\mu_1}\{\phi_{\mu_1}, H_c\}, \tag{100}$$

where $M^{\mu\nu}$ is the matrix inverse of the Poisson bracket matrix of the primary second-class constraints, $\{\phi_\mu, \phi_\nu\}$. Substituting $v_c^{\nu_1}$ for v^{ν_1} in (98) gives a more refined expression for the dynamics

$$\mathbf{X_H^1} := \frac{\partial}{\partial t} + \{-, H_c\}^* + v^{\mu_0}\{-, \phi_{\mu_0}\} \tag{101}$$

where a new structure has been introduced, the Dirac bracket. By the definition, the Dirac bracket will be

$$\{A,B\}^* := \{A,B\} - \{A,\phi_{\mu_1}\}M^{\mu_1\nu_1}\{\phi_{\nu_1},B\} \tag{102}$$

Notice that the evolutionary operator (101) can be alternatively rewriting (taking into account the fulfilment of the primary constraints) as

$$\mathbf{X}^{\mathbf{1}}_{\mathbf{H}} := \frac{\partial}{\partial t} + \{-,H_c^*\} + v^{\mu_0}\{-,\phi_{\mu_0}\} \tag{103}$$

with H_c^* a new canonical hamiltonian defined by

$$H_c^* := H_c - \{H_c,\phi_{\mu_1}\}M^{\mu_1\nu_1}\phi_{\nu_1} \tag{104}$$

making the use of the Dirac bracket, as we have said before, unnecessary.

Demonstration V.2

To prove the lemma (4.1), we are going to use the eq.(104). For that, let us compute, first of all, $M^{\mu\nu}$. Thus,

$$\{\phi_\mu,\phi_\nu\} = \begin{pmatrix} \{p_\alpha,p_\beta\} & \{p_\alpha,q_{\dot\beta}\} \\ \{q_{\dot\alpha},p_\beta\} & \{q_{\dot\alpha},q_{\dot\beta}\} \end{pmatrix}_{l\times l} = \begin{pmatrix} 0_{\alpha\beta} & -\delta_{\alpha\dot\beta} \\ \delta_{\dot\alpha\beta} & \{f_{\dot\alpha},f_{\dot\beta}\} \end{pmatrix}_{l\times l} \tag{105}$$

where α and β are indices that indicate the number of equations for each one, $\{-,-\}$ is the Poisson bracket and l indicates the dimension of the matrix that, in this case, is $2n$. Now, applying the definition of inverse matrix, $AA^{-1} = I_{l\times l}$, where I is the identity matrix of dimension $l\times l$, then

$$M^{\mu\nu} = (\{\phi_\mu,\phi_\nu\})^{-1} = \begin{pmatrix} \{f_\alpha,f_\beta\} & \delta_{\alpha\dot\beta} \\ -\delta_{\dot\alpha\beta} & 0_{\dot\alpha\dot\beta} \end{pmatrix}_{l\times l}. \tag{106}$$

So,

$$\begin{aligned} &\{H_c,\phi_\mu\}M^{\mu\nu}\phi_\nu = \\ &= (\{H_c,p_\alpha\},\{H_c,q_{\dot\alpha}\}) \begin{pmatrix} \{f_\alpha,f_\beta\} & \delta_{\alpha\dot\beta} \\ -\delta_{\dot\alpha\beta} & 0_{\dot\alpha\dot\beta} \end{pmatrix}_{l\times l} \begin{pmatrix} p_\beta \\ q_{\dot\beta} \end{pmatrix} \\ &= \{H_c,p_\alpha\}\{f_\alpha,f_\beta\}p_\beta - \{H_c,q_\alpha\}p_\alpha + \{H_c,p_\alpha\}q_\alpha. \end{aligned} \tag{107}$$

Finally, we get that

$$\begin{aligned} H_c^* &= H_c - \{H_c,p_\alpha\}\{f_\alpha,f_\beta\}p_\beta + \{H_c,q_\alpha\}p_\alpha - \{H_c,p_\alpha\}q_\alpha \\ &= H_c - \frac{\partial H_c}{\partial y_\alpha}\{f_\alpha,f_\beta\}p_\beta - \frac{\partial H_c}{\partial \pi_{y_\alpha}}p_\alpha - \{H_c,f_\alpha\}p_\alpha - \frac{\partial H_c}{\partial y_\alpha}y_\alpha + \frac{\partial H_c}{\partial y_\alpha}f_\alpha. \end{aligned} \tag{108}$$

Now, taking the derivative respect to one of the variables, we obtain that

$$\frac{\partial H_c^*}{\partial x_i} = \frac{\partial H_c}{\partial x_i} - \frac{\partial H_c}{\partial y_\alpha}\frac{\partial}{\partial x_i}\left(\{f_\alpha,f_\beta\}\right)p_\beta - \{H_c,\frac{\partial f_\alpha}{\partial x_i}\}p_\alpha + \frac{\partial H_c}{\partial y_\alpha}\frac{\partial f_\alpha}{\partial x_i} \tag{109}$$

and if we evaluate the eq.(109) on the second-class constraints' surface, we find that

$$\left.\frac{\partial H_c^*}{\partial x_i}\right|_{sc} = \left.\frac{\partial H_c}{\partial x_i}\right|_{cs} + \left.\frac{\partial H_c}{\partial y_\alpha}\right|_{cs}\frac{\partial f_\alpha}{\partial x_i}. \tag{110}$$

On the other hand, defining the $\bar{H}$ Hamiltonian as (27)

$$\bar{H}_c(x,\pi_x) = H_c(x_i,\pi_{x_i},\pi_{y_i}=0,y_i=f_i(x,\pi_x)) = H_c(x_i,\pi_{x_i},\pi_{y_i},y_i)|_{cs}\,, \tag{111}$$

its Hamilton's equation is

$$\frac{\partial \bar{H}_c}{\partial x_i} = \frac{\partial H_c}{\partial x_i}\bigg|_{cs} + \frac{\partial H_c}{\partial y_\alpha}\bigg|_{cs}\frac{\partial f_\alpha}{\partial x_i}. \tag{112}$$

Therefore, if we compare the eq.(110) with the eq.(112), we conclude that[3]

$$\frac{\partial H_c^*}{\partial x_i}\bigg|_{sc} \approx \frac{\partial \bar{H}_c}{\partial x_i} \tag{113}$$

they have the same Hamilton's equation and, as a result, the same equation of motion.

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[3] Notice that the demonstration for the momenta are exactly the same.

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