# Energy-momentum tensor for the electromagnetic field in a dispersive medium

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**Abstract**

The main objetive of this work is to present a mathematical procedure to obtain the field equations, the canonical energy-momentum tensor and the Belinfante-Rosenfeld tensor for non-local Lagrangians. This procedure is based on two steps: the first step is to convert the non-local Lagrangian into an order-$n$ Lagrangian which depends on any order derivatives of the field. The second step is to extend the order-$n$ to infinity and sum the formal series from the outcomes [1][2][3]. As the Lagrangian is transformed into a series, the equation of motion and Noether theorem can be derived as in [4]. Finally, this formalism is applied to classical electrodynamics for dispersive media.

## The field equations and the action principle

Consider a Lagrangian for a vector field in Minkowski spacetime, $A_a(x)$ where $a = 1, ..., 4$, that depends on the field derivatives up to the $n$-th order; the action is

$$S = \int_{\mathcal{V}} \mathcal{L}(A_a, A_{a;b}, ..., A_{a;b_1,...,b_n}) \mathrm{d}^4x \tag{1}$$

where the subindices after a semicolon mean partial derivative. The field equations can be obtained from $\delta\mathcal{S} = 0$ for field variation $\delta A_a$ such that $\delta A_{a;c_1,...,c_k} = 0$ for $0 \leq k \leq n$ on the boundary of $\mathcal{V}$. Applying the usual variational calculus, the following field equations are obtained

$$\Pi^a = 0, \qquad \text{with} \qquad \Pi^a \equiv \sum_{k=0}^{n} (-1)^k \partial_{b_1...b_k} \left( \frac{\partial \mathcal{L}}{\partial A_{a;b_1...b_k}} \right). \tag{2}$$

## Noether theorem for higher order Lagrangians

Assume that the Lagrangian is invariant by Poincaré transformations

$$\mathcal{L}(A_a, A_{a;b}...A_{a;b_1...b_n}) = \mathcal{L}(A'_a, A'_{a;b}, ..., A'_{a;b_1...b_n}). \tag{3}$$

The infinitesimal Poincaré transformation acts on coordinates as

$$x'^a = x^a + \delta x^a, \qquad \delta x^a = \epsilon^a + \omega^a_b x^b \qquad \text{where} \qquad \omega_{ab} + \omega_{ba} = 0 \tag{4}$$

and $A'_a(x') = A_a(x) - \omega^b_a A_b(x)$. Following the scheme as in the derivation of the field equations, the conserved current is obtained:

$$J^a = \epsilon^b \mathcal{T}_b{}^a + \frac{1}{2}\omega^{bc} \mathcal{J}_{bc}{}^a \tag{5}$$

with

$$\mathcal{T}_b{}^a = \sum_{k=0}^{\infty} \Pi^{d|c_1...c_k a} A_{d;c_1...c_k b} - \mathcal{L}\delta^a_b \qquad \text{and} \qquad \mathcal{J}_{bc}{}^a = 2x_{[c}\mathcal{T}_{b]}{}^a + \mathcal{S}_{bc}{}^a \tag{6}$$

where $\mathcal{S}^{bca} = P^{[bc]a} + Q^{[bc]a}$ and with

$$P^{bca} = 2\sum_{k=0}^{\infty} \Pi^{c|ad_1...d_k} A^b{}_{;d_1...d_k} \qquad \text{and} \qquad Q^{bca} = 2\sum_{k=0}^{\infty} k\ \Pi^{d|acd_1...d_{k-1}} A_d{}^{;b}{}_{d_1...d_{k-1}}. \tag{7}$$

## Minkowski electrodynamics for dispersive media

The electromagnetic field produced by a distribution of free charge and current in a medium is ruled by Maxwell equations. For a homogeneous isotropic dispersive medium, the simplest linear constitutive relation is $\mathbf{D} = \varepsilon * \mathbf{E}$ and $\mathbf{H} = \mu^{-1} * \mathbf{B}$ where $\varepsilon$ and $\mu$ respectively are the dielectric and magnetic functions. Because the constitutive relations are non-local, we postulate the following non-local action integral

$$S(x) = \frac{1}{4}\int \mathrm{d}^4x \int \mathrm{d}^4y F_{ab}(x)\hat{M}^{abcd}(y)F_{cd}(x-y) \tag{8}$$

where $F_{ab} = A_{b;a} - A_{a;b}$ and

$$\hat{M}^{abcd}(y) = (2\pi)^{-2}\left[\hat{m}(y)\hat{\eta}^{a[c}\hat{\eta}^{d]b} + 2\hat{\varepsilon}(y)u^{[a}\hat{\eta}^{b][c}u^{d]}\right] \tag{9}$$

where $\hat{\eta}^{ab} = \eta^{ab} + u^a u^b$ is the projector onto the hyperplane orthogonal to $u^b$, $\hat{m}(x)$ and $\hat{\varepsilon}(x)$ are the Fourier transform of $\mu(\omega,k)^{-1}$ and $\epsilon(\omega,k)$ respectively. A way of dealing with this sort of non-local Lagrangian consists in transforming it into an infinite order one by replacing $F_{ed}(x-y)$ with its Taylor expansion around $y = 0$

$$F_{ed}(x-y) = \sum_{k=0}^{\infty} \frac{y^{c_1}...y^{c_k}}{k!}(-1)^k F_{ed;c_1...c_k}(x) \tag{10}$$

This leads to the Lagrangian

$$\mathcal{L} = \sum_{k=0}^{\infty} M^{abcd|c_1...c_k} A_{a;b} A_{e;dc_1...c_k} \tag{11}$$

where the coefficients are

$$M^{abcd|c_1...c_k} = \frac{(-1)^k}{k!}\int \mathrm{d}^4y M^{abcd}(y) y^{c_1}...y^{c_k}. \tag{12}$$

To derive the field equations and its stress-energy tensor, the expressions (2) and (6) can be used. Therefore, the field equations are

$$\partial_b H^{ab} = 0 \qquad \text{where} \qquad H^{ab} = \hat{M}^{abcd} * F_{cd} \tag{13}$$

and the stress-energy tensor is

$$\mathcal{T}_b{}^a = \frac{1}{2}H^{ae}A_{e;b} - \frac{1}{4}\delta^a_b F_{ef}H^{ef} - \frac{1}{2}\int \mathrm{d}^4y\ \hat{M}^{efcd}(y)\frac{\partial}{\partial y^f}\int_0^1 \mathrm{d}\lambda\ y^a F_{cd}(\mathrm{X} - \lambda y)A_{e;b}(\mathrm{X}) \tag{14}$$

where $\mathrm{X} = x + (1-\lambda)y$.

The Belinfante-Rosenfeld tensor is defined as,

$$\Theta^{ba} = \mathcal{T}^{ab} + \partial_c \mathcal{W}^{cab} \tag{15}$$

where

$$\mathcal{W}^{cab} = \frac{1}{2}\Delta^{cab}_{klm}\left(P^{klm} + Q^{klm}\right) \qquad \text{with} \qquad \Delta^{cab}_{klm} = \delta^c_{[k}\delta^a_{l]}\delta^b_m + \delta^c_{[k}\delta^b_{l]}\delta^a_m - \delta^a_{[k}\delta^b_{l]}\delta^c_m \tag{16}$$

After some algebra, we obtain that

$$\begin{aligned}\Theta^{ba} = {} & \frac{1}{2}H^{ca}F_c{}^b - \frac{1}{4}\eta^{ab}F_{ed}H^{ed} + \frac{1}{2}F_{ed}\left[\hat{M}^{edh[b} * F_h^{a]} + \hat{M}^{edh(b;a)} * A_h + \frac{1}{2}\left(y^b\hat{M}^{edhf}\right) * F_{hf}{}^{;a}\right] \\ & - \frac{1}{2}\int \mathrm{d}^4y\hat{M}^{hfed}(y)\frac{\partial}{\partial y^f}\int_0^1 \mathrm{d}\lambda y^{(a}G^{b)}{}_{edh}(x,y,\lambda)\end{aligned} \tag{17}$$

where $G^b{}_{edh}(x,y,\lambda)$ is a function of the fields and their derivatives until second order in the spacetime points $\mathrm{X}$ and $x - \lambda y$.

The Lagrangian for a non-dispersive medium is a particular case with $\hat{M}^{abcd}(y) = M^{abcd}\delta^4(y)$, being $M^{abcd} = \text{constant}$. Therefore, applying the last condition to (17), we obtain

$$\Theta^{ba} = H^{ca}F_c{}^b - \frac{1}{4}\eta^{ab}F_{ed}H^{ed} \tag{18}$$

and the energy-momentum tensor for Minkowski electrodynamics is recovered.

## Conclusions

We show that, through the procedure of converting the non-local Lagrangian into an infinite series, the field equations, the canonical stress-energy tensor and the Belinfate-Rosenfeld tensor can be obtained for the case of electrodynamics for dispersive media. Moreover, applying the condition for non-dispersive media $\hat{M}^{abcd}(y) = M^{abcd}\delta^4(y)$, the Minkowski electrodynamics is recovered.

## References

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## Acknowledgements

Funding for this work was partially provided by the Spanish MINCIU and ERDF (project ref. RTI2018-098117-B-C22).