Remarkable equations. Fractional Operators

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1 Preliminar

The definition of fractional operators usually resorts to Laplace transforms and convolution. Since the "natural" arena for the latter seems to be the space of generalised functions with support on a half-line, we will first pinpoint some useful notions extracted from ref. [2] §10.2 to 10.4.

The inverse Laplace transform:

$\mathcal{D}'_+$ is the class of all distributions $f \in \mathcal{D}'$ such that $\text{supp}\, f \cap \mathbb{R}^- = \emptyset$,

$\mathcal{S}$ is the class of all functions $f \in \mathcal{C}^\infty$ such that decay faster than $|x|^{-n}$ for all $n \in \mathbb{Z}^+$,

$\mathcal{S}'$ is the class of all generalized functions acting on $\mathcal{S}$ and

$\mathcal{D}'_+(a)$ is the class of generalized functions $f \in \mathcal{D}'_+$ such that $\forall \sigma > a\,, \quad e^{-\sigma t} f \in \mathcal{S}'_+$. It is a convolution algebra

For any $f(t) \in \mathcal{D}'_+(a)$, the Laplace transform $\tilde{f}(s) = \mathcal{L}[f]$ is defined in VLAD(10.7) and is analytic in the half-plane $\mathbb{R}\text{e}(s) > a$.

The correspondence $f(t) \leftrightarrow \tilde{f}(s)$ is one-to-one. However, not all functions that are analytic in this half-plane are Laplace transforms of $f(t) \in \mathcal{D}'_+(a)$. The following property delimitates the range of $\mathcal{L}$.

Definition 1 *$H(a)$ is the subclass of functions $F(s)$ which:*

1(a) *are analytic in the half-plane $\mathbb{R}\text{e}(s) > a$ and*

1(b) *for any positive ε and $\sigma_0 > a$, there exist $C_\varepsilon(\sigma_0) \geq 0$ and $m = m(\sigma_0) \geq 0$ such that*

$$|F(s)| \leq C_\varepsilon(\sigma_0)\, e^{\varepsilon\,\mathbb{R}\text{e}(s)}\, (1 + |s|^m)\,, \qquad \mathbb{R}\text{e}(s) > \sigma_0 \tag{1}$$

$H(a)$ is an algebra with the usual product of functions.

Theorem 1 *$f(t) \in \mathcal{D}'_+(a)$ if, and only if, $\tilde{f}(s) \in H(a)$.*
Under these conditions, $\forall\, b \leq a\,,\;\; \sigma > \sigma_0 > a\,,\;\; k \in \mathbb{Z}\,,\quad k > m(\sigma_0) + 1\,,$

$$f(t) = \frac{1}{2\pi i} \left(\frac{\mathrm{d}}{\mathrm{d}t} - b\right)^k \int_{\sigma + i\mathbb{R}} \mathrm{d}s\, \frac{e^{st}}{(s-b)^k}\, \tilde{f}(s) \tag{2}$$

the rhs being independent of the choices of b, σ and k.

This is more general than the standard Mellin's integral [3] because it applyes when $\tilde{f}(\sigma + i\omega t)$ is not summable, which is remedied by dividing by $(\sigma + i\omega t)^k$. Depending on whether t is positive or negative, we can close the infinite line $\sigma + i\mathbb{R}$ with either the left or the right half-circles and, using the analyticity of the integrand, we can write the formula (2) as

$$f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} - b\right)^k \left\{\frac{\theta(t)}{2\pi i} \int_{\mathcal{C}} \mathrm{d}s \, \frac{e^{st}}{(s-b)^k} \, \tilde{f}(s)\right\} \tag{3}$$

where $\mathcal{C}$ is any simple path enclosing all singularities of $\tilde{f}(s)$ plus b. The rhs of the above formula includes one term with the prefactor $\theta(t)$ plus constant coefficients times the derivatives of $\delta(t)$ up to the order $k-1$, namely

$$f(t) = \theta(t)\, f_0(t) + \sum_{j=1}^{k-1} p_j \, \delta^j(t) \tag{4}$$

where

$$f_0(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} \mathrm{d}s \, e^{st} \, \tilde{f}(s) \, .$$

A remarkable property for $f(t) \in \mathcal{D}'_+(a)$ is that $\mathcal{L}\left[f^{(m)}(t)\right] = s^m \, \tilde{f}(s)$.

2 On the equation (2.16) in JHEP0208:008(2018)

To solve the equation

$$\sqrt{\partial_t^2 + 1}\, \phi(t) = 0 \tag{5}$$

we should first specify in which space we are seeking the solutions $\phi(t)$ and what the operator $\mathbf{T} := \sqrt{\partial_t^2 + 1}$ means.

As the operator's definition will involve Laplace transforms, the "natural" function space is $\mathcal{D}'_+(0)$, and we shall define $\mathbf{T}$ as the composition of functionals

$$\phi(t) \in \mathcal{D}'_+(0) \xrightarrow{\mathcal{L}} \tilde{\phi}(s) \in H(0) \xrightarrow{\sqrt{s^2+1}} \tilde{\phi}(s)\sqrt{s^2+1} \xrightarrow{\mathcal{L}^{-1}} \mathbf{T}\phi(t) \tag{6}$$

To ensure that the operations are well defined, we should prove that $\tilde{\phi}(s)\,\sqrt{s^2+1} \in H(0)$, i.e. it fulfills condition 1(b) above. Now, we have that $|\sqrt{s^2+1}| \le 1 + |s|$ and, as $\tilde{\phi}(s) \in H(0)$,

$$|\tilde{\phi}(s)| \le C_\varepsilon(\sigma_0)\, e^{\varepsilon\, \mathbb{R}\mathrm{e}(s)} \; (1 + |s|^m) \; , \qquad\qquad \mathbb{R}\mathrm{e}(s) > \sigma_0$$

Therefore

$$|\tilde{\phi}(s)| \cdot |\sqrt{s^2+1}| \le C_\varepsilon(\sigma_0)\, e^{\varepsilon\, \mathbb{R}\mathrm{e}(s)} \; (1 + |s|^m) \; (1 + |s|)$$

It can be easily proved that, for $x > 0$, $(1 + x^m)(1 + x) \le 4(1 + x^{m+1})$; which substituted above yields

$$|\tilde{\phi}(s)| \cdot |\sqrt{s^2+1}| \le 4\, C_\varepsilon(\sigma_0)\, e^{\varepsilon\, \mathbb{R}\mathrm{e}(s)} \; \left(1 + |s|^{m+1}\right)$$

and, by definition 1, $\tilde{\phi}(s)\,\sqrt{s^2+1} \in H(0)$. □

Applying Theorem 1,

$$\mathbf{T}\phi(t) := \frac{1}{2\pi i}\left(\frac{\mathrm{d}}{\mathrm{d}t} - b\right)^k \int_{\sigma+i\mathbb{R}} \mathrm{d}s\, \frac{e^{st}\sqrt{s^2+1}}{(s-b)^k}\, \tilde{\phi}(s) \tag{7}$$

for any appropriated b, k and σ .

Similarly, we can prove that $\sqrt{s^2+1} \in H(0)$, with $C_\varepsilon(\sigma_0) = 1$ and $m(\sigma_0) = 1$ and, by Theorem 1, its inverse Laplace transform is (we take $k = 3 > 1+1$, $b = 0$ and $\sigma > 0$)

$$f(t) = \frac{1}{2\pi i}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\int_{\sigma+i\mathbb{R}} \mathrm{d}s\, \frac{e^{st}\sqrt{s^2+1}}{s^3} = \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \mathrm{d}s\, e^{st}\sqrt{s^2+1}$$

where we have used the property of differentiation under the integral sign. (Notice that the prefactor $\theta(t)$ is implied by the integral along $\sigma + i\mathbb{R}$.)

$$\frac{\theta(t)}{2\pi i}\int_{\sigma+i\mathbb{R}} \mathrm{d}s\, e^{st}\sqrt{s^2+1} = \theta(t)\,(\partial_t^2+1)\left[\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \mathrm{d}s\, \frac{e^{st}}{\sqrt{s^2+1}}\right] = \theta(t)\,(\partial_t^2+1)J_0(t)$$

which, using the Bessel equation and the recurrence relations, leads to

$$f(t) = -\frac{\theta(t)}{t}\,\dot{J}_0(t) = \frac{\theta(t)}{t}\,J_1(t) \tag{8}$$

Using the Faltung theorem, the equation (5) becomes $f * \phi = 0$. According to Vladimirov §10.5(e), the operator $f*$ has an inverse $f^{-1}*$ if, and only if, $\dfrac{1}{\tilde{f}(s)} \in H(0)$. In our case, $a = 0$, and

$$\frac{1}{\sqrt{s^2+1}} = \mathcal{L}\left[\theta(t)\, J_0(t)\right]$$

As $\theta(t)\, J_0(t)$ is continuous and bounded, and its support is $[0,\infty[$, it belongs in $\mathcal{D}'_+(0)$; therefore, its Laplace transform is in $H(0)$.

As a consequence the equation $f * \phi = 0$, as an equation in $\mathcal{D}'_+(0)$, has the unique solution $\phi = 0$.

How does this match with ref. [1]?

It seems that [1] only considers **functions** $\phi(t)$, $t \in \mathbb{R}^+$ such that $e^{-\sigma t}\,\phi(t)$ is summable $\forall\,\sigma > 0$. The equation

$$\sqrt{\partial_t^2+1}\,\phi(t) = 0\,, \qquad t \in \mathbb{R}^+ \tag{9}$$

is fulfilled by any function $\phi(t)$ in $\mathcal{C}^m([0,\infty[)$ having an associated distribution $[\phi] = \theta(t)\,\phi(t) \in \mathcal{D}'_+(0)$ such that

$$\sqrt{\partial_t^2+1}\,[\phi] = P\,, \qquad \text{where} \qquad P = \sum_{j=0}^{n} a_j\, \delta^{(j)}(t) \tag{10}$$

is what, for the sake of brevity, we shall call a δ-polynomial.

Denoting by ΔP the subspace of δ-polynomials in $\mathcal{D}'_+(0)$, equation (9) amounts to

$$\sqrt{\partial_t^2+1}\,[\phi] \in \Delta P \tag{11}$$

It is not a homogeneous equation, and therefore it is not contradictory that equation (5) —in the sense of distributions— has only the trivial solution, whereas equation (9) —in the sense of functions in $\mathbb{R}^+$— has infinitely many. The equation in the sense of functions means, in fact, an infinity of equations in the sense of distributions.

The solution $[\phi] \in \mathcal{D}'_+(0)$ of (10) is

$$[\phi] = P * \mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+1}}\right) = P * [\theta(t)\, J_0(t)] = \theta(t) \sum_{j=0}^{n} p_j\, J_0^{(j)}(t) + \sum_{0 \le k < j \le n} p_j\, J_0^{(k)}(0)\, \delta^{(j-1-k)}(t) \tag{12}$$

The general solution of (9) is the associated function in $\mathbb{R}^+$, i. e. the regular part of $[\phi]$, namely

$$\phi(t) = \sum_{j=0}^{n} p_j\, J_0^{(j)}(t)\,, \qquad t \in \mathbb{R}^+ \tag{13}$$

which depends on infinitely many parameters (recall that n is not fixed).

Is $(\partial_t^2+1) = \sqrt{\partial_t^2+1} \cdot \sqrt{\partial_t^2+1}$ **?**

It is indeed, provided that (7) is used as the definition for the action of $\sqrt{\partial_t^2+1}$ on both $\mathbb{R}^+$ functions and distributions. However this does not mean that any solution $\phi(t)$ of (9) is also a solution of $(\partial_t^2+1)\,\phi(t) = 0$. Indeed

$$\sqrt{\partial_t^2+1}\,\phi(t) = 0 \quad \Leftrightarrow \quad \sqrt{\partial_t^2+1}\,[\phi] \in \Delta P \quad \Rightarrow \quad (\partial_t^2+1)\,[\phi] \in \sqrt{\partial_t^2+1}\,(\Delta P)$$

Now $\sqrt{\partial_t^2+1}\,(\Delta P)$ is not a subset of ΔP and, as a rule, $(\partial_t^2+1)\,\phi \neq 0$.

The paradox lies in the fact that we take (13) as the solution of equation (9), but the solution of the homogeneous equation (in the sense of distributions) is (12) and contains extra δ-terms, which are the cause that $(\partial_t^2+1)\,\phi \neq 0$.

3 On $D^\alpha D^\beta = D^{\alpha+\beta}$

In section 2.4.4 in ref.[1] it is said that the Liouville definition may lead to unexpected results, e.g. that the above equality does not hold for non-integer α and β. This is true in the sense of functions, as in ref[4], but it is false in the sense of distributions (ref. [2] §7.8).

Define the generalized functions $f_\alpha \in \mathcal{D}'_+$ (convolution algebra):

$$f_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}\,\theta(t)\,, \qquad \forall\, \alpha > 0 \tag{14}$$

It is easy to prove that, $\forall \alpha > 1$, $\quad f'_\alpha(t) = f_{\alpha-1}(t) \quad$ and by iteration it follows that, $\forall n \in \mathbb{Z}^+$ and $\forall \alpha > n$,

$$f_\alpha^{(n)}(t) = f_{\alpha-n}(t) \tag{15}$$

This property allows extending the definition (14) to non-positive α's, namely

$$f_\alpha(t) := f^{(n)}_{\alpha+n}(t)\,, \qquad \text{for any } n \text{ such that} \quad \alpha + n > 0 \tag{16}$$

In particular, $f_1(t) = \theta(t)$ and $f_0(t) = f_1'(t) = \delta(t)$.

Proposition. $f_\alpha * f_\beta = f_{\alpha+\beta}\,, \quad \forall \alpha, \beta\,.$

Proof:[2] For positive α and β, we apply the definition (14)

$$f_\alpha * f_\beta = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\,\theta(t)\int_0^t \mathrm{d}t'\,t'^{\alpha-1}(t-t')^{\beta-1} = \ldots = \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\,\theta(t) = f_{\alpha+\beta}$$

If one or both indices, α or β, is non-positive, we use (15) to have that, for any positive integers n, msuch that α and $\beta + m$ are positive, $f_\alpha = f^{(n)}_{\alpha+n}$ and $f_\beta = f^{(m)}_{\beta+m}$, then

$$f_\alpha * f_\beta = f^{(n)}_{\alpha+n} * f^{(m)}_{\beta+m} = (f_{\alpha+n} * f_{\beta+m})^{(m+n)} = f^{(m+n)}_{\alpha+\beta+m+n} = f_{\alpha+\beta}$$

Fractional derivative [Riemann-Liouville operator] Given a generalized function $\varphi \in \mathcal{D}'_+$ and a real α, the *fractional derivative* of order α is

$$D^\alpha \varphi := f_{-\alpha} * \varphi \tag{17}$$

An example

It is said in [1], §2.4.4 that " *. . . considering the function* $\phi(t) = t^{-1/2}$ *, which satisfies* $D^{1/2}\phi = 0$ *, while* $D^1\phi = \partial_t\phi = -\frac{1}{2}\,t^{-3/2} \neq 0\ldots$ "

It must be remarked that the natural arena for the Riemann-Liouville fractional derivative to act is $\mathcal{D}'_+$, and therefore the $\phi(t)$ should actually be $\phi(t) = t^{-1/2}\,\theta(t) = \Gamma(\frac{1}{2})\,f_{1/2}$. Hence

$$D^{1/2}\phi = f_{-1/2} * \phi = f_{-1/2} * \left[\Gamma(\frac{1}{2})\,f_{1/2}\right] = \Gamma(\frac{1}{2})\,f_0(t) = \Gamma(\frac{1}{2})\,\delta(t) \neq 0$$

By the way, $D^{1/2}\phi$ is a δ-polynomial.

In turn,

$$\begin{aligned} D^{1/2}\left(D^{1/2}\phi\right) &= D^{1/2}\left[\Gamma(\frac{1}{2})\,\delta(t)\right] = \Gamma(\frac{1}{2})\,\left[f_{-1/2} * f_0\right] = \Gamma(\frac{1}{2})\,f_{-1/2} \\ &= \Gamma(\frac{1}{2})\,f'_{1/2} = \Gamma(\frac{1}{2})\,\frac{\mathrm{d}}{\mathrm{d}t}\left]\frac{t^{-1/2}}{\Gamma(1/2)}\,\theta(t)\right] = \ldots = \frac{\mathrm{d}\phi}{\mathrm{d}t} \end{aligned}$$

contrary to what is said in [1], §2.4.4

References

[1] Barnaby N and Kamran N, “Dynamics with Infinitely Many Derivatives: The Initial Value Problem”, arXiv:0709.3968 [hep-th]

[2] Vladimirov V S *Equations of Mathematical Physics* (MIR, Moscow, 1984)

[3] Weinberger H F *A First Course in Partial Differential Equations* (Dover, 1995)

[4] Kilbas A, Srivastava H and Trujillo J *Theory and Applications of Partial Differential Equations* (North-Holland, Elsevier, 2006