Ph.D. thesis defence:

# NONLOCAL LAGRANGIAN FORMALISM

C. Heredia Pimienta

Supervisor: **Prof. Josep Llosa**

University of Barcelona, Departament de Física Quàntica i Astrofísica

Institute of Cosmos Sciences (ICCUB)

www.carlosherediapimienta.com

# Thesis structure

- Local Lagrangians: { Mechanics, Fields }
- Nonlocal Lagrangians: { Mechanics, Fields }

# Nonlocal Lagrangians - Mechanics

- Definitions: What does
    - the extended kinematic space
    - a nonlocal Lagrangian
    - the time evolution

  means?
- The principle of least action
    - The nonlocal Euler-Lagrange equations
    - Two ways of coordinating the extended kinematic space
    - The extended dynamic space
- Nonlocal total derivative

# Nonlocal Lagrangians - Mechanics

- **THE NOETHER THEOREM**
  - The energy function
- The Hamiltonian formalism
  - The Legendre transformation

Example: Nonlocal harmonic oscillator

# Motivations

- Consider the following Lagrangian:

$$L([q],t) = q(t)\,(G*q)_{(t)} = q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(t-\sigma)q(\sigma)$$

How do we get the EOMs or the momenta?

$$= q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(-\sigma)q(\sigma+t)$$

and then apply Taylor... $q(\sigma+t) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} q^{(n)}(t)$

$$= \sum_{n=0}^{\infty} a_n\, q(t) q^{(n)}(t) \qquad \left(a_n := \int_{\mathbb{R}} \mathrm{d}\sigma \frac{\sigma^n G(-\sigma)}{n!}\right)$$

But, I have to integrate by parts infinite times...

Let us write it as

$$L([q],t) = q(t)\,(G*q)_{(t)} = \sum_{n=0}^{\infty} a_n\, q(t) q^{(n)}(t)$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} q(t) q^{(n)}(t)$$

then we apply the Ostrogradski formalism to get:

**EOMs:**

$$\lim_{N\to\infty} \sum_{n=0}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \left[\frac{\partial L}{\partial q^{(n)}}\right] = 0$$

Differential equations and moments are no longer correctly defined.

**Momenta:**

$$p_n := \lim_{N\to\infty} \sum_{k=n+1}^{N} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-n-1} \frac{\partial L}{\partial q^{(k)}}$$

# Definitions

# Definitions

- **Def:** The class of all (possible) kinematic trajectories is the

*kinematic space* $\mathcal{K} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^s)$

$q^i(t) \quad (i = 1, \ldots, s)$

- **Def:** The *extended kinematic space* is $\mathcal{K}' = \mathcal{K} \times \mathbb{R}$

for time-dependent Lagrangians

- **Def:** A *nonlocal Lagrangian* is a real-valued functional on $\mathcal{K}'$

$$([q^i], t) \in \mathcal{K}' \longrightarrow L([q^i], t) \in \mathbb{R}$$

where $[q^i]$ means functional dependence on the whole $q^i(\sigma)$

**Ex:** $L([q], t) = q(t)\,(G * q)_{(t)} = q(t) \int_{\mathbb{R}} \mathrm{d}\sigma\, G(t - \sigma) q(\sigma)$

# Definitions

- **Def:** We define the *time evolution operator* $T_t$ as

$$([q^i],0) \xrightarrow{T_t} ([T_t q^i],t) \qquad \text{where} \qquad T_t q^i(\sigma) = q^i(\sigma + t)$$

**Ex:**

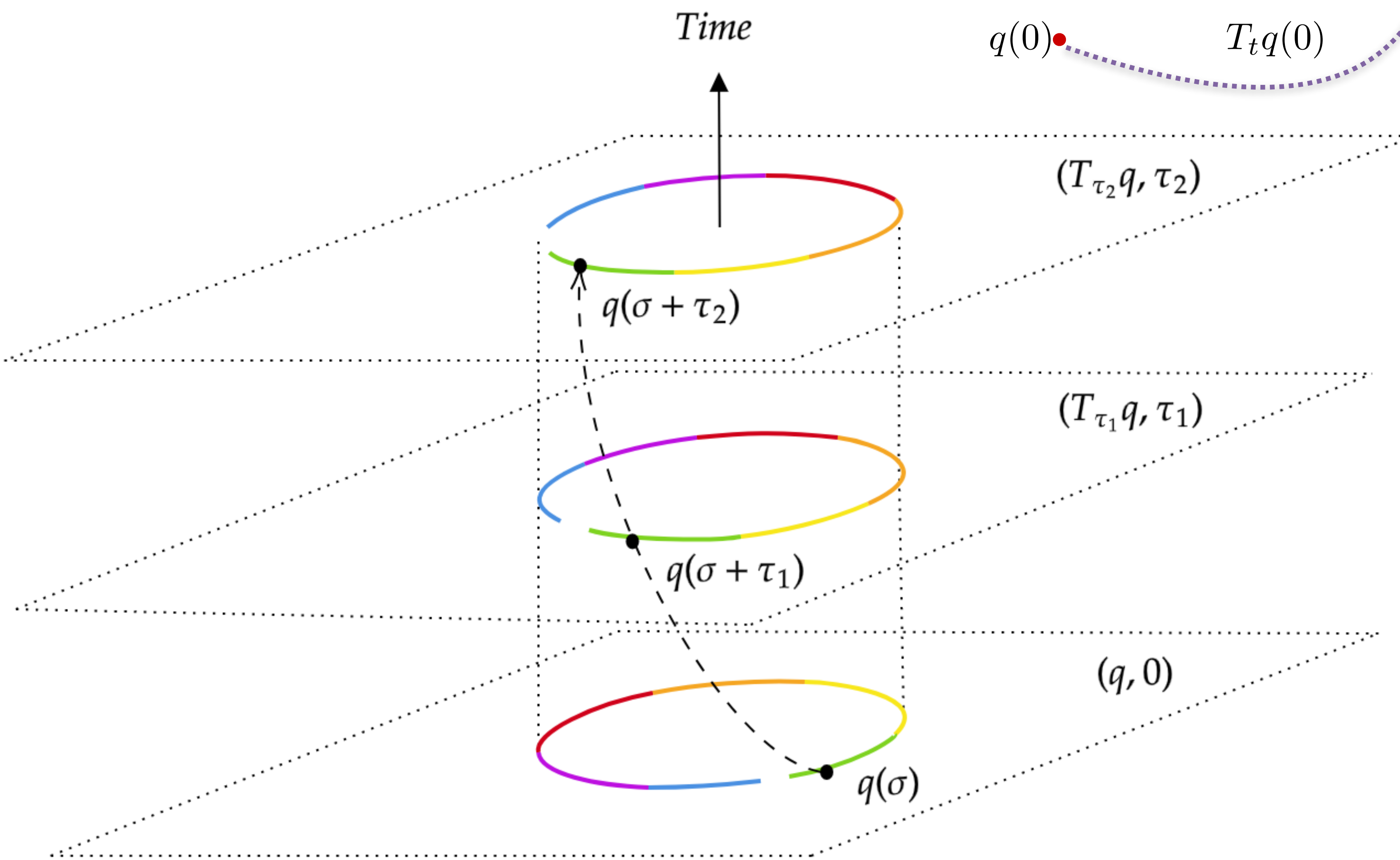
$$L(T_t q) = q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(t-\sigma)q(\sigma) = q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(-\sigma)q(\sigma+t)$$

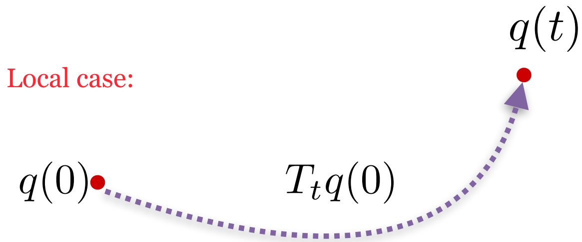
$$= T_t q(0)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(-\sigma) T_t q(\sigma)$$

- **Obs:** It is possible to establish a one-to-one correspondence between the "infinite-order" Ostrogradsky formalism and the nonlocal one via formal Taylor's series (FTS)

$$\left(\left\{q^{i,(r)}(t)\right\}_{r\in\mathbb{N}}, t\right) \longleftrightarrow ([q^i],t) \qquad \text{with} \qquad q^i(\sigma+t) = \sum_{k=0}^{\infty}\frac{\sigma^k}{k!}q^{i,(k)}(t)$$

The curve $([T_t q], t)$ visually is

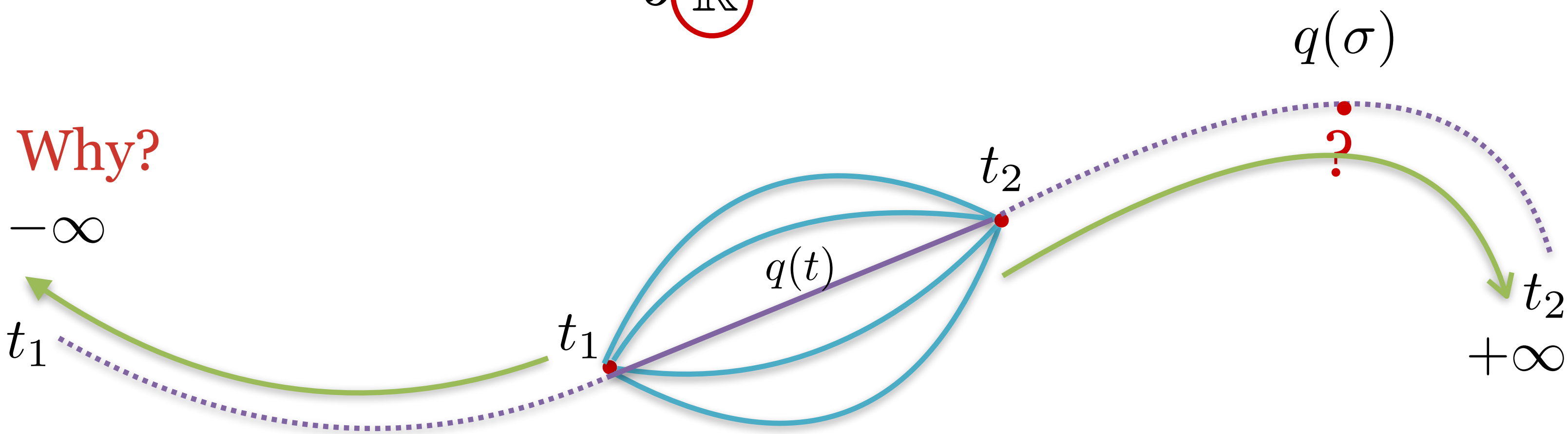


# Principle of least action

- **Def:** We define the *nonlocal action integral* as

$$S(q) := \int_{\mathbb{R}} \mathrm{d}t \, L\left(T_t q, t\right)$$



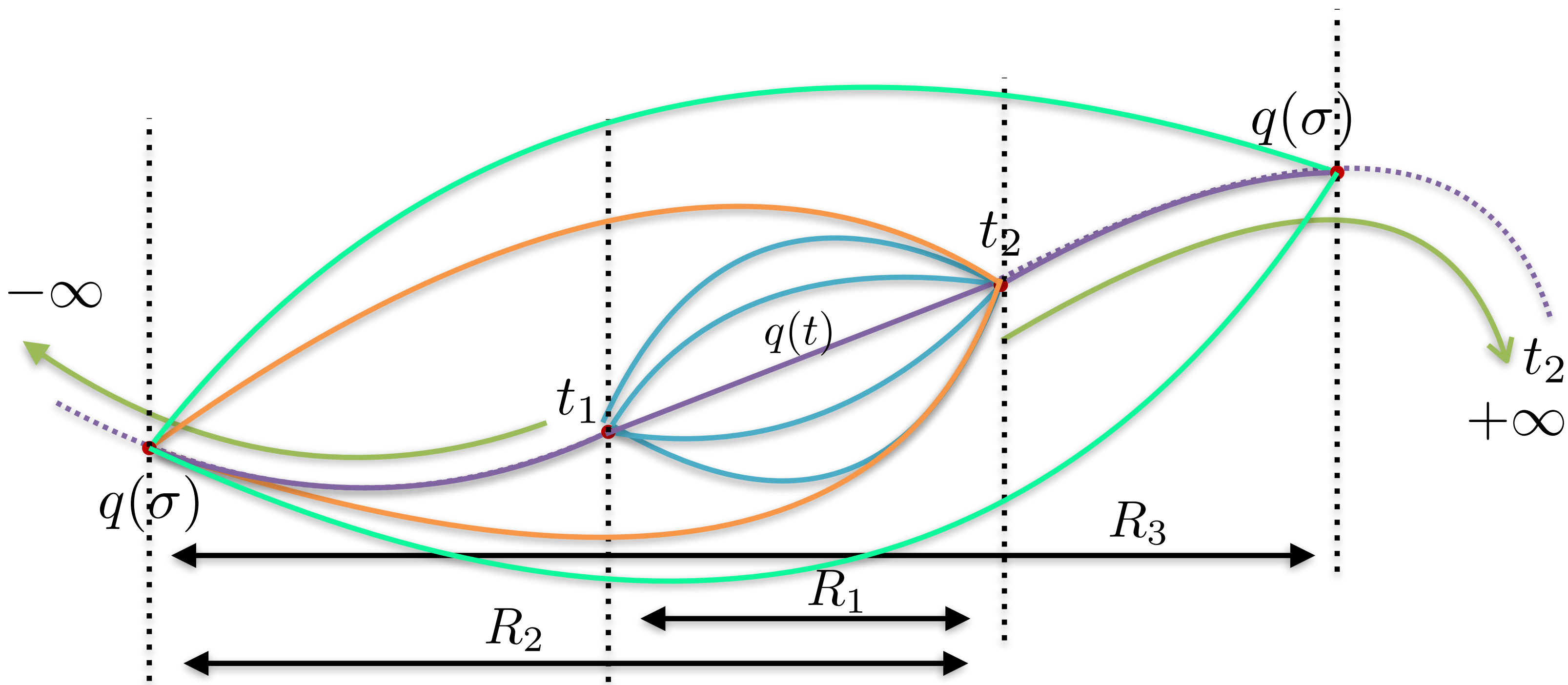

where $t \in [t_1, t_2]$ but, in nonlocal theories, $q(\sigma)$ with $\sigma \in \mathbb{R}$

If it is unbounded, could it be infinity?

It could be. For this reason, we introduce:

the 1-parameter family of finite action integrals

$$S(q,R) = \int_{|t|\leq R} \mathrm{d}t\, L(T_t q, t)\,, \qquad \forall R \in \mathbb{R}^+$$



- The principle of least action reads

$$\lim_{R\to\infty} \delta S(q,R) \equiv \lim_{R\to\infty} \int_{|t|\leq R} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\sigma \, \frac{\delta L(T_t q, t)}{\delta q(\sigma)} \, \delta q(\sigma)$$

$$= \int_{\mathbb{R}} \mathrm{d}\sigma \, \delta q(\sigma) \int_{\mathbb{R}} \mathrm{d}t \, \frac{\delta L(T_t q, t)}{\delta q(\sigma)} = 0$$

$\forall \delta q(\sigma)$ of compact support.

It can be shown that, with a standard Lagrangian of the first order, we can recover the Euler-Lagrange equations

- As long as the limit exists, the nonlocal Euler-Lagrange equations

$$\lambda(q,t,\sigma) := \frac{\delta L(T_t q, t)}{\delta q(\sigma)}$$

$$\psi(q,\sigma) := \int_{\mathbb{R}} \mathrm{d}t \, \lambda(q,t,\sigma)$$

$\forall \sigma \in \mathbb{R}$

Cauchy's problem!

are: $$\psi(q,\sigma) = 0$$

- We might coordinate a point $z \in \mathcal{K}'$ in two different ways:

$$\textbf{(a)} \quad \begin{array}{rcl} \mathcal{K}' & \longrightarrow & \mathcal{K} \times \mathbb{R} \\ z & \longmapsto & (\tilde{q}, t) \end{array} \qquad \textbf{(b)} \quad \begin{array}{rcl} \mathcal{K}' & \longrightarrow & \mathcal{K} \times \mathbb{R} \\ z & \longmapsto & (q, t) \end{array}$$

where $q(\sigma) = T_t \tilde{q}(\sigma) = \tilde{q}(\sigma + t)$

We shall refer to:

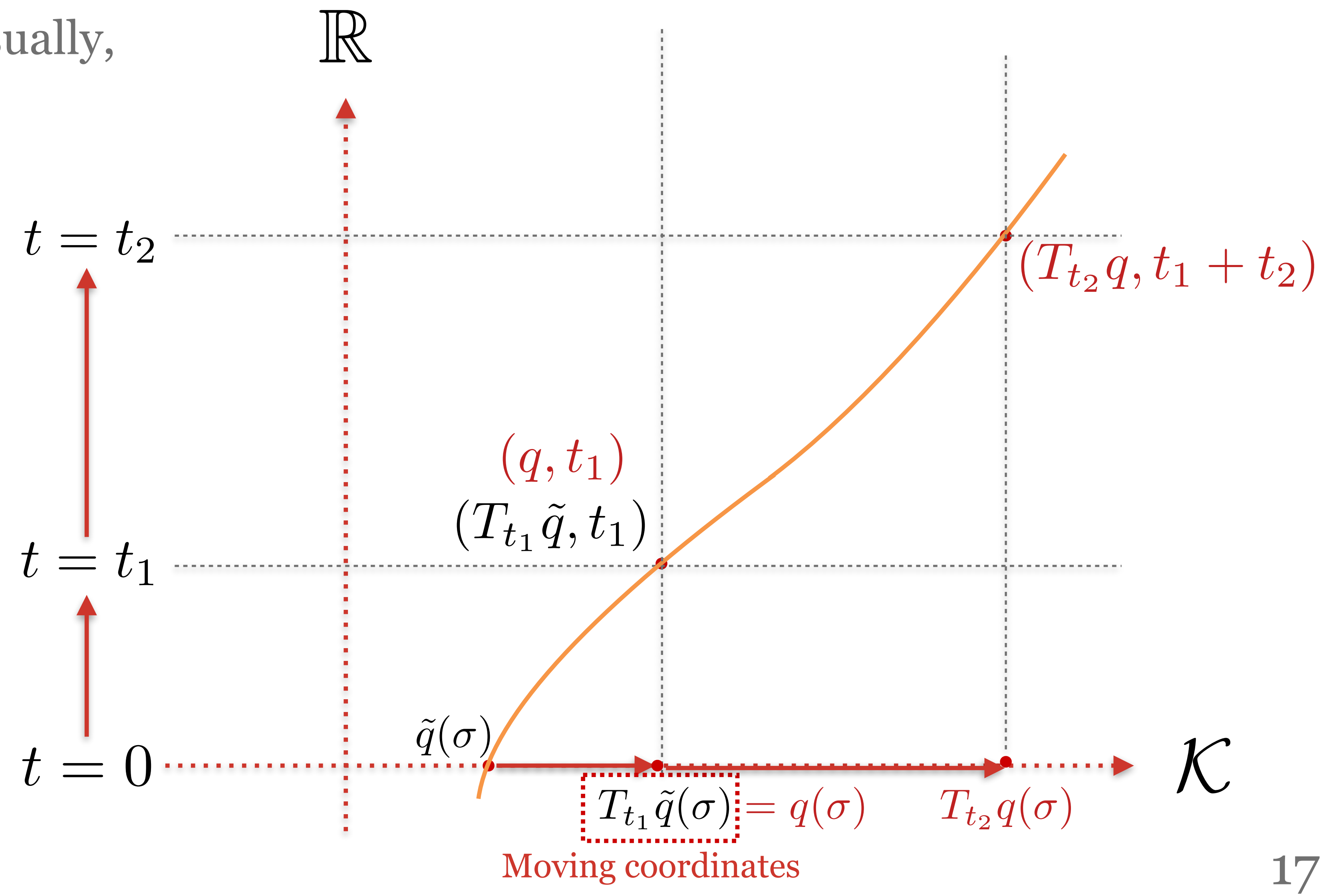
$(\tilde{q}, t)$ as static coordinates

$(q, t)$ as moving *coordinates*

- Why? Because the time evolution expression in each of these coordinate systems is:

$$\textbf{(a)} \quad (\tilde{q}, t) \xrightarrow{T_\tau} (\tilde{q}, t + \tau) \qquad \textbf{(b)} \quad (q, t) \xrightarrow{T_\tau} (T_\tau q, t + \tau)$$

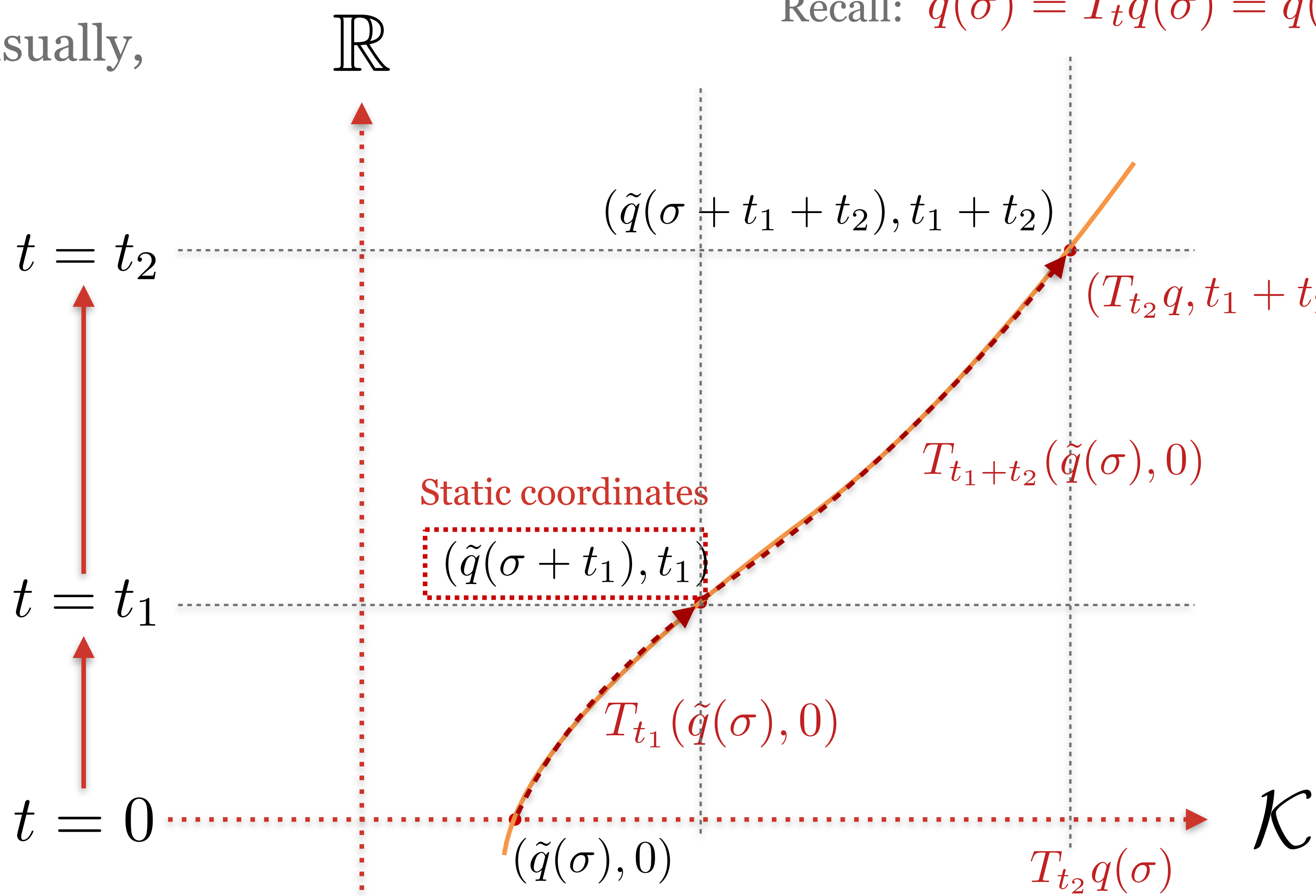
Visually,


$\mathbb{R}$
$t = t_2$
$t = t_1$
$t = 0$
$(T_{t_2} q, t_1 + t_2)$
$(q, t_1)$
$(T_{t_1} \tilde{q}, t_1)$
$\tilde{q}(\sigma)$
$\mathcal{K}$
$T_{t_1} \tilde{q}(\sigma) = q(\sigma)$
$T_{t_2} q(\sigma)$
Moving coordinates

# Principle of least action

Recall: $q(\sigma) = T_t \tilde{q}(\sigma) = \tilde{q}(\sigma + t)$

Visually,



# Principle of least action

- Therefore, going back again to the nonlocal Euler-Lagrange equations,

$$\psi(q,\sigma) = 0 \qquad \text{where} \qquad \psi(q,\sigma) := \int_{\mathbb{R}} \mathrm{d}t\, \lambda(q,t,\sigma)$$

$$\lambda(q,t,\sigma) := \frac{\delta L(T_t q, t)}{\delta q(\sigma)}$$

From slide 10: $$L(T_t q) = q(t) \int_{\mathbb{R}} \mathrm{d}\sigma\, G(-\sigma) q(\sigma + t)$$

We see that they are obtained in the context of *static coordinantes*, limited to trajectories

$$(T_t q, t) \in \mathcal{K}' \quad \text{starting at} \quad (q, 0)$$

# Principle of least action

- Therefore, they should be better understood:

$$\psi(\tilde{q},\sigma)=0 \quad \text{with} \quad \psi(\tilde{q},\sigma):=\int_{\mathbb{R}} \mathrm{d}t\,\lambda(\tilde{q},t,\sigma)$$

$$\lambda(\tilde{q},t,\sigma):=\frac{\delta L(T_t\tilde{q},t)}{\delta\tilde{q}(\sigma)}$$

- In moving coordinates, but why? I want my trajectory to start at $(\tilde{q},t)$ and not at $(\tilde{q},0)$

$$L(T_t\tilde{q},t) \xrightarrow[(T_\tau q,\, t+\tau)]{q=T_t\tilde{q}} L(T_\tau q,t+\tau)$$

$$\Psi(q,t,\sigma)=0 \quad \text{with} \quad \Psi(q,t,\sigma)\equiv\psi(\tilde{q},t+\sigma)$$

Notice that: $\psi(\tilde{q},\sigma)=\Psi(q,0,\sigma)$

- The infinitesimal time generator $\mathbf{D}$ acts on them

$$\mathbf{D}\psi(\tilde{q},\sigma) = \partial_t \psi(\tilde{q},\sigma)$$

$$\mathbf{D}\Psi(q,t,\sigma) = \left[\frac{\partial \Psi(T_\epsilon q, t+\epsilon, \sigma)}{\partial \epsilon}\right]_{\epsilon=0}$$

- Indeed, the nonlocal Euler-Lagrange equations are stable under time evolution:

$$\mathbf{D}\psi(\tilde{q},\sigma) = 0 \quad \text{or} \quad \mathbf{D}\Psi(q,t,\sigma) = 0$$

- **Def:** The extended dynamic space $\mathcal{D}'$ is the class of all dynamic trajectories, namely, those $(q, t)$ that satisfy the nonlocal Euler-Lagrange equations. So, $\mathcal{D}' \subset \mathcal{K}'$

- **Obs:** As a rule, there are no general theorems of existence and uniqueness for integrodifferential equations.

Therefore, coordinating the dynamic space with the initial conditions will depend on the case.

- An alternative view of the nonlocal Euler-Lagrange equations:

They are the constraints (in implicit form) that define the extended dynamic space as a submanifold of the extended kinematic space.

# Nonlocal total derivative

# Nonlocal total derivative

- **Obs:** It is well known that if the Lagrangian is total derivative, the EOMs are identities.
- But, is it true for nonlocal ones?
- Let's see it with an example. Consider:

The kernel of the integral operator vanishes at $|x| \to \infty$

$$L(T_t\tilde{q}, t) = \dot{\tilde{q}}(t)\,(G * \tilde{q})_{(t)}$$

To find the total derivative, we need to solve the following equation:

$$L(T_t\tilde{q}, t) = \frac{\mathrm{d}W(T_t\tilde{q}, t)}{\mathrm{d}t}$$

Indeed, a particular solution is:

$$W(T_t\tilde{q}, t) = \int_{\mathbb{R}} \mathrm{d}\sigma\,[\theta(\sigma) - \theta(\sigma - t)]\,L(T_\sigma\tilde{q}, \sigma)$$

# Nonlocal total derivative

- The nonlocal action integral is

$$S(\tilde{q},R)=\int_{|t|\leq R}\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}\tau\,\dot{\tilde{q}}(t)\,\tilde{q}(\tau)\,G(t-\tau)$$

whose nonlocal Euler-Lagrange equations are

$$\psi(\tilde{q},\sigma)\equiv(\dot{G}_{|+}*\tilde{q})_{(\sigma)}=0$$

with $\dot{G}(x)_{|+}=\dot{G}(x)+\dot{G}(-x)$

Therefore, only the odd part of the kernel matters:

$$\dot{G}(t-\tau)=\dot{G}(\tau-t)\quad\text{or}\quad G(t-\tau)=-G(\tau-t)$$

It's an odd function!

# Nonlocal total derivative



Recall: $L(T_t\tilde{q},t) = \dfrac{\mathrm{d}W(T_t\tilde{q},t)}{\mathrm{d}t}$

Now, we compute

$$W(T_t\tilde{q},t) = \tilde{q}(t)\,(G * \tilde{q})_{(t)} - \int_{\mathbb{R}^2} \mathrm{d}\rho\,\mathrm{d}\sigma\,[\theta(\rho) - \theta(\rho - t)]\,\dot{G}(\rho - \sigma)\,\tilde{q}(\rho)\,\tilde{q}(\sigma)$$

To check if the nonlocal Euler-Lagrange equation vanish,

$$\frac{\delta L(T_t\tilde{q},t)}{\delta\tilde{q}(\sigma)} = \dot{\delta}(t-\sigma)\,(G * \tilde{q})_{(t)} + \dot{\tilde{q}}(t)G(t-\sigma)$$

then

$$\psi(\tilde{q},\sigma) = -(\dot{G} * \tilde{q})_{(\sigma)} + \int_{\mathbb{R}} \mathrm{d}t\,\dot{\tilde{q}}(t)\,G(t-\sigma)$$

$$= \int_{\mathbb{R}} \mathrm{d}t\,\dot{\tilde{q}}(t)\,[G(t-\sigma) - G(\sigma - t)] \neq 0 \text{ !!!!}$$

because the kernel is not an even function!

**The nonlocal Euler-Lagrange equations are not identically zero!**

# Nonlocal total derivative

**What condition does the non-local total derivative need to satisfy to remain the EOMs unchanged?**

- Let us search a sufficient condition on $W(T_t\tilde{q}, t)$
- The family action integrals are

$$S(\tilde{q}, R) = \int_{|t|\leq R} \mathrm{d}t\, L(T_t\tilde{q}, t) = \int_{|t|\leq R} \mathrm{d}t\, \frac{\mathrm{d}W(T_t\tilde{q}, t)}{\mathrm{d}t}$$

$$= W(T_R\tilde{q}, R) - W(T_{-R}\tilde{q}, -R)$$

# Nonlocal total derivative

- The variational principle yields the following equations

$$\psi(\tilde{q},\sigma) = \lim_{R\to\infty} \frac{\delta S(\tilde{q},R)}{\delta \tilde{q}(\sigma)}$$
$$= \lim_{R\to\infty} \left[ \frac{\delta W(T_R\tilde{q},R)}{\delta \tilde{q}(\sigma)} - \frac{\delta W(T_{-R}\tilde{q},-R)}{\delta \tilde{q}(\sigma)} \right] \equiv 0$$

- So, a sufficient condition is:

$$\boxed{\lim_{R\to\pm\infty} \frac{\delta W(T_R\tilde{q},R)}{\delta \tilde{q}(\sigma)} = 0}$$

- **Obs:** if it is a Noether symmetry: $\delta L = \frac{\mathrm{d}W(T_t\tilde{q},t)}{\mathrm{d}t}$

  but with $W(T_t\tilde{q},t)$ satisfying this condition.

# Noether's theorem

## Noether's Theorem for nonlocal Lagrangians

- Consider the infinitesimal transformations:

$$t'(t) = t + \delta t(t) \qquad \text{and} \qquad \tilde{q}'(t) = \tilde{q}(t) + \delta\tilde{q}(t)$$

The nonlocal Lagrangian will transform so that it leaves the nonlocal action integral invariant, namely,

$$L'(T_{t'}\tilde{q}', t') = \left|\frac{\mathrm{d}t}{\mathrm{d}t'}\right| L(T_t\tilde{q}, t)$$

Thus, if $[t_0, t_1]$ is a time interval and $[t'_0, t'_1]$ is the transformed one,

$$\int_{t'_0}^{t'_1} \mathrm{d}t'\, L'(T_{t'}\tilde{q}', t') = \int_{t_0}^{t_1} \mathrm{d}t\, L(T_t\tilde{q}, t)$$

# Noether's theorem

- As $S'(\tilde{q}', t'_0, t'_1) = S(\tilde{q}, t_0, t_1)$ :

$$\int_{t'_0}^{t'_1} \mathrm{d}t\, L'(T_t\tilde{q}', t) - \int_{t_0}^{t_1} \mathrm{d}t\, L(T_t\tilde{q}, t) = 0$$

where the dummy variable $t'$ is replaced with $t$

- Given those infinitesimal transformations, the first integral can be approximated to the leading order as

$$\int_{t'_0}^{t'_1} \mathrm{d}t\, L'(T_t\tilde{q}', t) = \int_{t_0}^{t_1} \mathrm{d}t \left\{ L'(T_t\tilde{q}', t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(T_t\tilde{q}, t)\delta t\right] \right\}$$

Therefore,

$$\int_{t_0}^{t_1} \mathrm{d}t \left\{ L'(T_t\tilde{q}', t) - L(T_t\tilde{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(T_t\tilde{q}, t)\delta t\right] \right\} = 0$$

# Noether's theorem

- We say that a transformation is a Noether symmetry if

$$L'(T_t\tilde{q}',t) = L(T_t\tilde{q}',t) + \frac{\mathrm{d}}{\mathrm{d}t}W(T_t\tilde{q}',t)$$

where $W(T_t\tilde{q}',t)$ satisfies the previous asymptotic condition.

- These quantities become up to the leading order:

$q'(t) = q(t) + \delta q(t)$

$\lambda(\tilde{q},t,\sigma)$

$$L(T_t\tilde{q}',t) \doteq L(T_t\tilde{q},t) + \int_{\mathbb{R}} \mathrm{d}\sigma \frac{\delta L(T_t\tilde{q},t)}{\delta\tilde{q}(\sigma)}\delta\tilde{q}(\sigma)$$

$$W(T_t\tilde{q}',t) = W(T_t\tilde{q},t) + \int_{\mathbb{R}} \mathrm{d}\sigma \frac{\delta W(T_t\tilde{q},t)}{\delta\tilde{q}(\sigma)}\delta\tilde{q}(\sigma)$$

Therefore,

$$L'(T_t\tilde{q}',t) = L(T_t\tilde{q},t) + \frac{\mathrm{d}}{\mathrm{d}t}W(T_t\tilde{q},t) + \int_{\mathbb{R}} \mathrm{d}\sigma\lambda(\tilde{q},t,\sigma)\delta\tilde{q}(\sigma)$$

- Let us introduce the last equation into

$$\int_{t_0}^{t_1} \mathrm{d}t \left\{ L'(T_t\tilde{q}',t) - L(T_t\tilde{q},t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(T_t\tilde{q},t)\delta t\right] \right\} = 0$$

- Thus,

$$\int_{t_1}^{t_2} \mathrm{d}t \left\{ \delta L + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\delta \dot{q}}\delta \dot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(q,\dot{q},t)\delta t\right] \right\}$$

Local case...

$$\int_{t_0}^{t_1} \mathrm{d}t \left\{ \int_{\mathbb{R}} \mathrm{d}\sigma \lambda(\tilde{q},t,\sigma)\,\delta\tilde{q}(\sigma) + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(T_t\tilde{q},t)\delta t + W(T_t\tilde{q},t)\right] \right\} = 0$$

Crucial point! How do you get the boundary terms and EOMs?

Recall: $\psi(\tilde{q},\sigma) = 0$ with $\psi(\tilde{q},\sigma) = \int_{\mathbb{R}} \mathrm{d}t\,\lambda(\tilde{q},t,\sigma)$

- Let us add and subtract $\int_{\mathbb{R}} \mathrm{d}\sigma \lambda(\tilde{q},\sigma,t)\delta\tilde{q}(t) = \psi(\tilde{q},t)\delta\tilde{q}(t)$

- We obtain:

$$\int_{t_0}^{t_1} \mathrm{d}t \left\{ \psi(\tilde{q},t)\delta\tilde{q}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(T_t\tilde{q},t)\delta t + W(T_t\tilde{q},t)\right] \right.$$

Is there a way to link them?

$$\left. + \int_{\mathbb{R}} \mathrm{d}\sigma \lambda(\tilde{q},t,\sigma)\delta\tilde{q}(\sigma) - \int_{\mathbb{R}} \mathrm{d}\sigma \lambda(\tilde{q},\sigma,t)\delta\tilde{q}(t) \right\} = 0$$

- After a suitable change of variable:

$$\int_{\mathbb{R}} \mathrm{d}\xi \; \left[\lambda(\tilde{q},t,t+\xi)\,\delta\tilde{q}(t+\xi) - \lambda(\tilde{q},t-\xi,t)\,\delta\tilde{q}(t)\right]$$

- Now, by using the following identity:

$$\lambda(\tilde{q},t,t+\xi)\,\delta\tilde{q}(t+\xi) - \lambda(\tilde{q},t-\xi,t)\,\delta\tilde{q}(t) =$$

$$= \int_0^1 \mathrm{d}\eta\, \frac{\partial}{\partial\eta}\left[\lambda(\tilde{q},t+(\eta-1)\xi,t+\eta\xi)\,\delta\tilde{q}(t+\eta\xi)\right]$$

$$= \xi \int_0^1 \mathrm{d}\eta\, \frac{\partial}{\partial t}\left[\lambda(\tilde{q},t+(\eta-1)\xi,t+\eta\xi)\,\delta\tilde{q}(t+\eta\xi)\right]$$

- We find:

$$\int_{t_0}^{t_1} \mathrm{d}t \left\{ \psi(\tilde{q},t)\delta\tilde{q}(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(T_t\tilde{q},t)\delta t + W(T_t\tilde{q},t) + U(T_t\tilde{q},t)\right] \right\} = 0$$

where

$$U(T_t\tilde{q},t) := \int_{\mathbb{R}} \mathrm{d}\xi\, \xi \int_0^1 \mathrm{d}\eta\, \lambda(\tilde{q}, t+(\eta-1)\xi, t+\eta\xi)\, \delta\tilde{q}(t+\eta\xi)$$

⋮ (After a bit of algebra...)

Using the analogy with the local case, this will be the Legendre transformation for nonlocal Lagrangians!

$$U(T_t\tilde{q},t) = \int_{\mathbb{R}} \mathrm{d}\rho\, \delta\tilde{q}(t+\rho)\, P(\tilde{q},t,\rho)$$ with

$$P(\tilde{q},t,\rho) := \int_{\mathbb{R}} \mathrm{d}\zeta\, \left[\theta(\rho)-\theta(\zeta)\right] \frac{\delta L(T_{t+\zeta}\tilde{q}, t+\zeta)}{\delta\tilde{q}(t+\rho)}$$

# Noether's theorem

Since the choice of the interval is completely arbitrary, we obtain:

- Noether's identity:

$$\psi(\tilde{q},t)\,\delta\tilde{q}(t)+\frac{\mathrm{d}}{\mathrm{d}t}\left[L\left(T_t\tilde{q},t\right)\delta t(t)+W\left(T_t\tilde{q},t\right)+U\left(T_t\tilde{q},t\right)\right]\equiv 0$$

- and Noether's conserved quantity:

Of course: $\left.\frac{\mathrm{d}}{\mathrm{d}t}J(T_t\tilde{q},t)\right|_{\psi=0}=0$

$$J(T_t\tilde{q},t):=L\left(T_t\tilde{q},t\right)\delta t(t)+W\left(T_t\tilde{q},t\right)+U\left(T_t\tilde{q},t\right)$$

We have just extended Noether's theorem for nonlocal Lagrangians.

# NTH: Energy function

# Energy function

- First of all, let us rewrite the Noether current in moving coordinates $q = T_t\tilde{q}$

  Then,

$$J(q,t) = L\,(q,t)\,\delta t(t) + W(q,t) + U(q,t)$$

  where

$$U(q,t) = \int_{\mathbb{R}} \mathrm{d}\rho\,\delta q(\rho)\,P(q,t,\rho)$$

  with

$$P(q,t,\rho) = \int_{\mathbb{R}} \mathrm{d}\zeta\ [\theta(\rho) - \theta(\zeta)]\ \frac{\delta L(T_\zeta q, t+\zeta)}{\delta q(\rho)}$$

- Consider the time-translation transformation

$$\delta t = \epsilon \quad \text{and} \quad \delta q = -\epsilon\,\dot{q}$$

# Energy function

- **Def:** The energy function -in moving coordinates- is

$$E(q) := -\epsilon^{-1} J(q) = -L\,(q) + \int_{\mathbb{R}} \mathrm{d}\rho\, \dot{q}(\rho)\, P(q,\rho)$$

with

$$P(q,\rho) := \int_{\mathbb{R}} \mathrm{d}\zeta\, \left[\theta(\rho) - \theta(\zeta)\right] \frac{\delta L(T_\zeta q)}{\delta q(\rho)}$$

where we have assumed that the nonlocal Lagrangian does not explicitly depend on time $W(q) = 0.$

Considering a first-order Lagrangian, one can recover the usual structure of the energy function.

$$E(q_0, \dot{q}_0) := \frac{\partial L_L(q_0, \dot{q}_0)}{\partial \dot{q}_0} \dot{q}_0 - L_L(q_0, \dot{q}_0)$$

# Hamiltonian formalism

# Hamiltonian formalism

- **Def:** The nonlocal Hamiltonian on the extended phase space $\Gamma' = \Gamma \times \mathbb{R}$ is defined -in moving coordinates- as follows:

$$H(q,\pi,t) = \int_{\mathbb{R}} \mathrm{d}\sigma\, \pi(\sigma)\, \dot{q}(\sigma) - L(q,t)$$

that is equipped with the following Poisson bracket

$$\{F,G\} = \int_{\mathbb{R}} \mathrm{d}\sigma \left( \frac{\delta F}{\delta q(\sigma)} \frac{\delta G}{\delta \pi(\sigma)} - \frac{\delta F}{\delta \pi(\sigma)} \frac{\delta G}{\delta q(\sigma)} \right)$$

- Hamilton's equations are

$$\begin{cases} \mathbf{X}_H q(\sigma) = \dot{q}(\sigma) \\ \mathbf{X}_H \pi(\sigma) = \dot{\pi}(\sigma) + \dfrac{\delta L(q,t)}{\delta q(\sigma)} \end{cases}$$

where $\mathbf{X}_H$ is the Hamiltonian vector field

$$\mathbf{X}_H = \partial_t + \int_{\mathbb{R}} \mathrm{d}\sigma \left( \dot{q}(\sigma) \frac{\delta}{\delta q(\sigma)} + \left[ \dot{\pi}(\sigma) + \frac{\delta L(q,t)}{\delta q(\sigma)} \right] \frac{\delta}{\delta \pi(\sigma)} \right)$$

- Hamilton's equations can be written in a more compact form by using the contact differential 2-form:

$$\Omega' = \Omega - \delta H \wedge \delta t \quad \text{with} \quad \Omega = \int_{\mathbb{R}} \mathrm{d}\sigma\, \delta\pi(\sigma) \wedge \delta q(\sigma)$$

(differential)

Namely,

$$i_{\mathbf{X}_H} \Omega' = 0$$

So far, this Hamiltonian system has nothing to do with the Lagrangian formulation previously presented.
How can we connect them?

- We can connect both through the injection:

$$(q,t) \in \mathcal{D}' \xrightarrow{j} (q,\pi,t) \in \Gamma'$$

where

$$\pi(\sigma) := P(q,t,\sigma)$$

It is the prefactor in the Noether conserved quantity

$\delta q(\sigma) P(q,t,\sigma)$

- **Def:** $j$ defines a one-to-one map from $\mathcal{D}'$ into $j(\mathcal{D}') \subset \Gamma'$ namely, a submanifold implicitly defined by the constraints:

$$\Psi\,(q,t,\sigma) = 0 \quad \text{and} \quad \Upsilon\,(q,\pi,t,\sigma) := \pi(\sigma) - P(q,t,\sigma) = 0$$

- Besides, $j^T \mathbf{D} = X_{\mathbf{H}}$

Consequently, the constraints are stable by the Hamiltonian flow.

# Hamiltonian formalism

- To translate the Hamiltonian formalism in $\Gamma'$ into $\mathcal{D}'$, we use the pullback $j^*$

Therefore, the contact form $\omega' \in \Lambda^2(\mathcal{D}')$ is $\omega' := j^*\Omega'$

$$\omega'(q,t) = \omega(q,t) - \delta h(q,t) \wedge \delta t$$

where

$$\omega(q,t) = \int_{\mathbb{R}} \mathrm{d}\sigma \, \delta P(q,t,\sigma) \wedge \delta q(\sigma)$$

It's close but it is not clear whether it is non-degenerate!

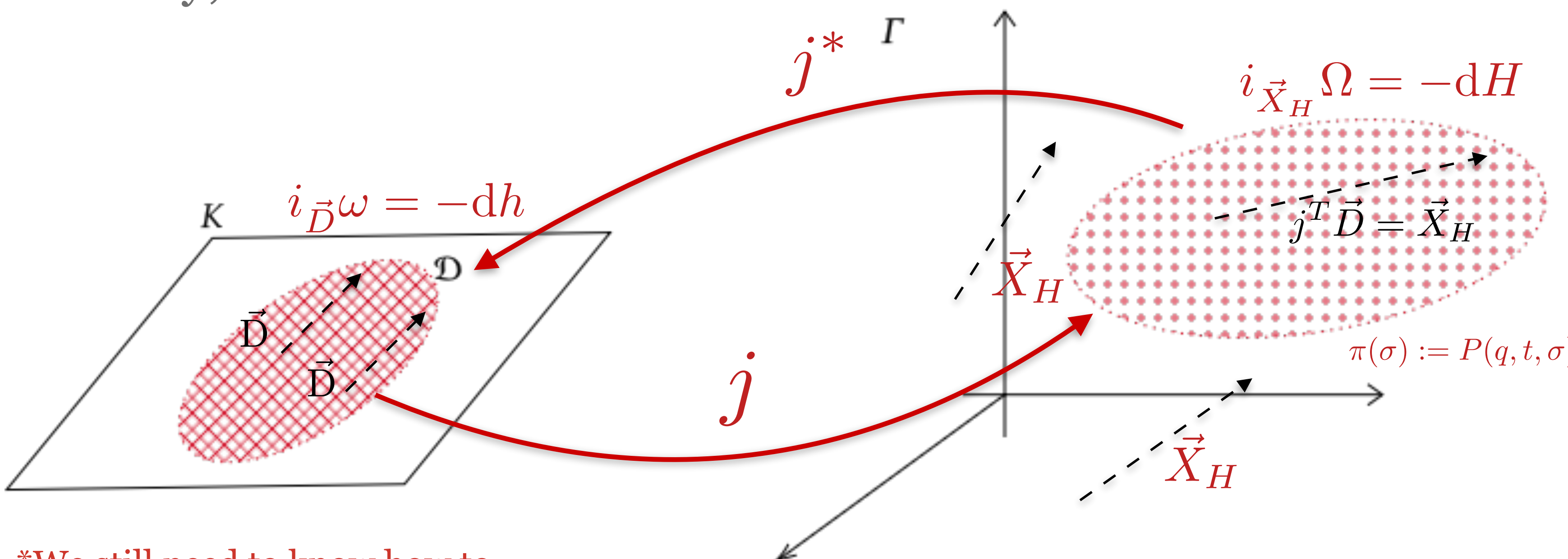
and, as $j^T \mathbf{D} = \mathbf{X}_H$, the Hamilton equations read

$$i_{\mathbf{X}_H}\Omega' = 0 \quad \xrightarrow{\;j^*\;} \quad i_{\mathbf{D}}\omega' = 0$$

Finally, the Hamiltonian in $\mathcal{D}'$ is $h := j^* H$

$$h(q,t) = \int_{\mathbb{R}} \mathrm{d}\sigma \, P(q,t,\sigma)\dot{q}(\sigma) - L(q,t)$$

Visually,




*We still need to know how to coordinate the dynamic space.

# Application: NL H.O.

Consider the following Lagrangian:

$K(\zeta) = e^{-|\zeta|}$

$$S(\tilde{q}, R) = \int_{|t| \leq R} \mathrm{d}t \left[ \frac{1}{2}\, \dot{\tilde{q}}^2(t) - \frac{\omega^2}{2}\, \tilde{q}^2(t) + \frac{g}{4}\, \tilde{q}(t) \int_{\mathbb{R}} \mathrm{d}\zeta\, K(\zeta)\, \tilde{q}(t-\zeta) \right]$$

Let us identify this nonlocal Lagrangian with our formalism, so

$$L\,(T_t\tilde{q}) = \frac{1}{2}\, \dot{\tilde{q}}^2(t) - \frac{\omega^2}{2}\, \tilde{q}^2(t) + \frac{g}{4}\, \tilde{q}(t) \int_{\mathbb{R}} \mathrm{d}\zeta\, K(\zeta)\, \tilde{q}(t-\zeta)$$

$$= \frac{1}{2}\, \underline{T_t}\dot{\tilde{q}}_0^2 - \frac{\omega^2}{2}\, \underline{T_t}\tilde{q}_0^2 + \frac{g}{4}\, \underline{T_t}\tilde{q}_0 \int_{\mathbb{R}} \mathrm{d}\zeta\, K(\zeta)\, \underline{T_t}\tilde{q}(-\zeta)$$

OK!

The nonlocal Euler-Lagrange equations are

$$\psi(\tilde{q}, \sigma) = -\ddot{\tilde{q}}(\sigma) - \omega^2 \tilde{q}(\sigma) + \frac{g}{2}\, (K * \tilde{q})_{(\sigma)} = 0$$

# Example: NL H.O.

To solve this integro-differential equation,

$$\ddot{\tilde{q}}(\sigma) + \omega^2 \tilde{q}(\sigma) - \frac{g}{2}\left(K * \tilde{q}\right)_{(\sigma)} = 0$$

$\frac{\mathrm{d}^2}{\mathrm{d}\sigma^2}$

$$\tilde{q}^{(iv)}(\sigma) + (\omega^2 - 1)\,\ddot{\tilde{q}}(\sigma) + (g - \omega^2)\tilde{q}(\sigma) = 0$$

The general solution is

$$\tilde{q}(\sigma) = \sum_{j=1}^{4} A^j\, e^{\sigma\, r_j}$$

where $r_j$ are the roots of the characteristic equation

$$r^4 + (\omega^2 - 1)r^2 + g - \omega^2 = 0$$

# Example: NL H.O.

Namely, the solution is

$$r_{\pm\pm} = \pm r_{\pm} \quad \begin{cases} r_{\pm} = \sqrt{\dfrac{1-\omega^2}{2} \pm \sqrt{\Delta}} \\ \Delta = \dfrac{(\omega^2+1)^2}{4} - g \end{cases}$$

Because $\tilde{q}(\sigma)$ is a solution, $(K * \tilde{q})$ must exist. Therefore, it implies:

$$|\mathbb{R}e(r_j)| < 1$$

Thus, the general solution is:

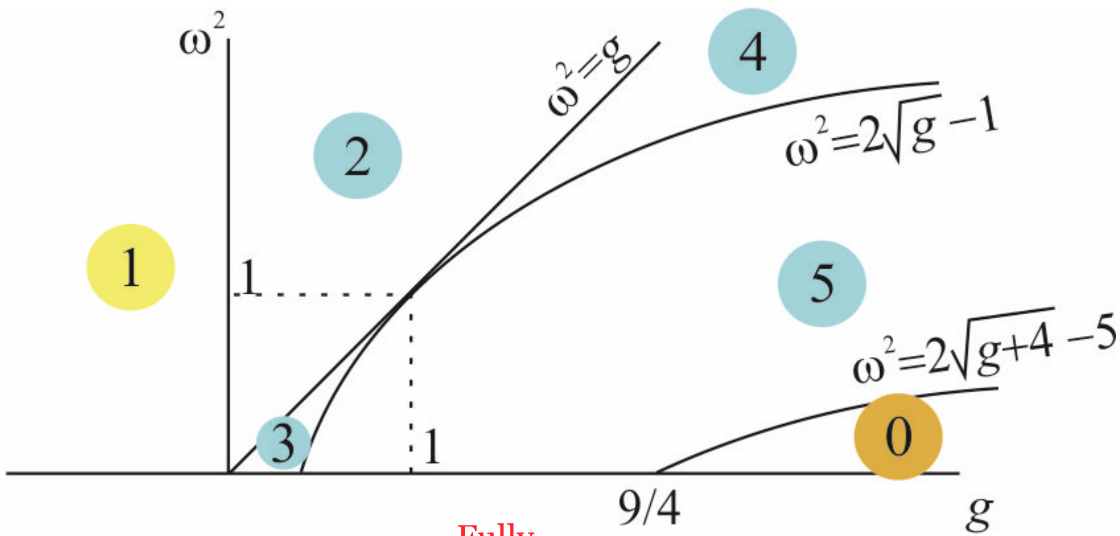
They are the coordinates of $\mathcal{D}'$

$$\tilde{q}(\sigma) = \sum_{j=1}^{4} A^j e^{\sigma\, r_j} \quad \text{with} \quad |\mathbb{R}e(r_j)| < 1$$

They are the parametric equations of $\mathcal{D}'$

# Example: NL H.O.

$$\tilde{q}(\sigma) = \sum_{j=1}^{4} A^j \, e^{\sigma \, r_j}$$




$|\mathbb{R}\mathrm{e}(r_j)| > 1!!$

Completely real roots — Fully imaginary Roots — Condition to be satisfied

| Region | | real $r_j$ | imaginary $r_j$ | $\lvert\mathbb{R}\mathrm{e}(r_j)\rvert < 1$ |
|---|---|---|---|---|
| 0 | $\omega^2 \leq 2\sqrt{g+4} - 5$ | 0 | 0 | 0 |
| 1 | $g \leq 0$ | 2 | 2 | 2 |
| 2 | $\omega^2 \geq g > 0$ | 2 | 2 | 4 |
| 3 | $2\sqrt{g} - 1 \leq \omega^2 < \min\{g, 1\}$ | 4 | 0 | 4 |
| 4 | $\max\{1, 2\sqrt{g} - 1\} < \omega^2 < g$ | 0 | 4 | 4 |
| 5 | $2\sqrt{g+4} - 5 < \omega^2 < 2\sqrt{g} - 1$ | 0 | 0 | 4 |

Region 1: $\mathrm{Dim}\, \mathcal{D}' = 2 + 1$

Region 4: $\mathrm{Dim}\, \mathcal{D}' = 4 + 1$

# Example: NL H.O.

Because we have a nonlocal time-independent Lagrangian,

$$L\left(T_t\tilde{q}\right) = \frac{1}{2}\,\dot{\tilde{q}}^2(t) - \frac{\omega^2}{2}\,\tilde{q}^2(t) + \frac{g}{4}\,\tilde{q}(t)\int_{\mathbb{R}} \mathrm{d}\zeta\, K(\zeta)\,\tilde{q}(t-\zeta)$$

the nonlocal Euler-Lagrange in moving coordinates are

$$\Psi(q,\sigma) = -\ddot{q}(\sigma) - \omega^2 q(\sigma) + \frac{g}{2}\,(K * q)_{(\sigma)} = 0$$

and the solution is obviously

$$q(\sigma) = \sum_{j=1}^{4} B^j\, e^{\sigma\, r_j} \qquad \text{with} \qquad |\mathbb{R}e(r_j)| < 1$$

Notice that: $q(0) = \sum_j B^j = \tilde{q}(t) = \sum_j A^j e^{r_j t} \implies \boxed{B^j = A^j e^{r_j t}}$

# Example: NL H.O.



Let us compute the momenta:

$$P(q,\sigma) = \int_{\mathbb{R}} \mathrm{d}\zeta\, [\theta(\sigma) - \theta(\zeta)]\, \frac{\delta L(T_\zeta q)}{\delta q(\sigma)}$$

$L(T_\zeta q)$

⋮ (After a bit of algebra...)

$$= \delta(\sigma)\, \dot{q}(\sigma) + \frac{g}{4}\theta(\sigma)(K * q)_{(\sigma)} - \frac{g}{4}\int_0^\infty \mathrm{d}\zeta\, K(\zeta - \sigma)\, q(\zeta)$$

$q(\sigma)$

⋮ (After a bit of algebra...)

$$= \sum_{j=1}^{4} B^j \left( r_j \delta(\sigma) + \frac{g}{4} \frac{e^{-|\sigma|}}{r_j + \mathrm{sign}(\sigma)} \right)$$

# Example: NL H.O.

Let us compute the (pre)symplectic form.

First of all, because our Lagrangian does not depend explicitly on time, $\omega' = \omega$

The (pre)symplectic form is

$$\omega'(q) = \int_{\mathbb{R}} \mathrm{d}\sigma \, \delta P(q,\sigma) \wedge \delta q(\sigma)$$

$\delta P(q,\sigma)$

$\delta q(\sigma)$

$\vdots$

(After a bit of algebra...)

$$= \left(\sum_{j=1}^{4} r_j \, \delta B^j\right) \wedge \left(\sum_{k=1}^{4} \delta B^k\right) + g \left(\sum_{j=1}^{4} \frac{\delta B^j}{1 - r_j^2}\right) \wedge \left(\sum_{k=1}^{4} \frac{r_k \, \delta B^k}{1 - r_k^2}\right)$$

Next,

$$= \left( \sum_{j=1}^{4} r_j \, \delta B^j \right) \wedge \left( \sum_{k=1}^{4} \delta B^k \right) + g \left( \sum_{j=1}^{4} \frac{\delta B^j}{1 - r_j^2} \right) \wedge \left( \sum_{k=1}^{4} \frac{r_k \, \delta B^k}{1 - r_k^2} \right)$$

The coordinantes $B^j$ are related to $q_0 \, , \, \dot{q}_0 \, , \ldots , \, \dddot{q}_0$ through

$$\sum_j r_j^n \, B^j = q_0^{(n)} \, , \qquad n = 0, \ldots, 3$$

Thus,

$$\omega = \underbrace{\delta \dot{q}_0}_{p_0} \wedge \delta q_0 + \frac{1}{g} \, \delta \underbrace{\left[ \omega^2 q_0 + \ddot{q}_0 \right]} \wedge \delta \underbrace{\left[ \omega^2 \dot{q}_0 + \dddot{q}_0 \right]}$$

$$\pi_0 := \frac{1}{\sqrt{g}} \left( \omega^2 \, q_0 + \ddot{q}_0 \right) \quad \xi_0 := \frac{1}{\sqrt{g}} \left( \omega^2 \, \dot{q}_0 + \dddot{q}_0 \right)$$

$$= \delta p_0 \wedge \delta q_0 + \delta \pi_0 \wedge \delta \xi_0$$

The non-vanishing elementary Poisson brackets

$$\{ q_0, p_0 \} = \{ \xi_0, \pi_0 \} = 1$$

For the Hamiltonian, we can procedure in the same way.

The Hamiltonian is

$$h(q) = \int_{\mathbb{R}} \mathrm{d}\sigma \, P(q,\sigma)\dot{q}(\sigma) - L(q)$$

$P(q,\sigma)$

$q(\sigma)$

(After a bit of algebra…)

$$= \frac{1}{2}\dot{q}_0^2 + \frac{\omega^2}{2} q_0^2 - \frac{g}{2} \sum_{j,k=1}^{4} B^j B^k \frac{1 + r_j r_k - r_j^2 - r_k^2}{(1 - r_j^2)(1 - r_k^2)}$$

(After a bit of algebra…)

The variables: $(q, p, \pi, \xi)$

$$= \frac{1}{2} p_0^2 + \frac{\omega^2}{2} q_0^2 - \sqrt{g}\, q_0 \, \pi_0 + \frac{1}{2} \pi_0^2 - \frac{1}{2} \xi_0^2$$

The sign is not defined, so the system is unstable? No!

# Conclusions

# Conclusions

We have seen that:

- We develop a new formalism to deal with nonlocal Lagrangians.

- We extend Noether's theorem for them.

  It gives us, by analogy with the local case, the definition of the Legendre transform.

- We propose a Hamiltonian formalism for them.

  Once we know how to coordinate the extended dynamic space, it provides us with the Hamiltonian and the symplectic form.

# Conclusions

Observations:

- All of the above we have extended to classical field theory.
- Some applications:
  - Nonlocal electrodynamics - Dispersive Media.
  - Nonlocal harmonic oscillator.
  - p-adic strings.
  - Non-commutative theories (work in progress).

## Dispersive Media:

$$M^{abcd}(x) = (2\pi)^{-2}\left[m(x)\,\hat{\eta}^{a[c}\hat{\eta}^{d]b} + 2\varepsilon(x)\,u^{[a}\hat{\eta}^{b][c}u^{d]}\right]$$

$$S(\tilde{A}, R) = \frac{1}{4}\int_{|x|\leq R} \mathrm{d}x\,\tilde{F}_{ab}(x)\left(M^{abcd} * \tilde{F}_{cd}\right)_{(x)}$$

## The (Belinfante-Rosenfeld) energy momentum tensor:

(for a wave package)

$$\mathcal{U} \approx \frac{1}{4}\,\mathbb{R}\mathrm{e}\left[\frac{\mathrm{d}(\varepsilon\omega)}{\mathrm{d}\omega}\,\underline{\tilde{\mathbf{E}}}\cdot\underline{\tilde{\mathbf{E}}}^{*} + \frac{\mu^{*}}{\mu}\,\frac{\mathrm{d}(\mu\omega)}{\mathrm{d}\omega}\,\underline{\tilde{\mathbf{H}}}\cdot\underline{\tilde{\mathbf{H}}}^{*}\right]$$

$$G^{i} \approx \frac{1}{4}\,\mathbb{R}\mathrm{e}\left[\frac{1}{\omega\mu}\,\frac{\mathrm{d}(\varepsilon\mu\omega^{2})}{\mathrm{d}\omega}\,\underline{\tilde{\mathbf{E}}}\times\underline{\tilde{\mathbf{B}}}^{*}\right]$$

$$S^{i} \approx \frac{1}{2}\,\mathbb{R}\mathrm{e}\left[\underline{\tilde{\mathbf{E}}}^{*}\times\underline{\tilde{\mathbf{H}}}^{*}\right]$$

New...

$$T^{ij} \approx \frac{1}{2}\,\mathbb{R}\mathrm{e}\left[\underline{\tilde{E}}^{*\,i}\,\underline{\tilde{D}}^{j} + \underline{\tilde{H}}^{i}\,\underline{\tilde{B}}^{*\,j} - \frac{1}{2}\left(\underline{\tilde{\mathbf{D}}}\cdot\underline{\tilde{\mathbf{E}}}^{*} + \underline{\tilde{\mathbf{H}}}\cdot\underline{\tilde{\mathbf{B}}}^{*}\right)\delta^{ij}\right]$$

Thank you!

Any question?