Noether's Theorem for Nonlocal Lagrangians

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Emmy Noether

- AMALIE EMMY **NOETHER**
  (1882-1935)
- Mathematician
  (Albert Einstein, Felix Klein and David Hilbert considered her one of the most brilliant minds.)
- Idea: Conservation laws derive from a more fundamental relationship: symmetries.

NOETHER'S THEOREM

"FOR EVERY CONTINUOUS SYMMETRY, THERE EXISTS A CONSERVED QUANTITY"

Noether's article

E. Noether,

- "Invariante Variationsprobleme,"
- Nachr. v. d. Ges. d. Wiss. zu Göttingen 1918, pp235-257

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und

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1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17

Int. Action:

$$\mathcal{S}([q]) = \int_{t_0}^{t_1} \mathrm{d}t \, L(q, \dot{q}, t)$$

+ the inf. transformations

$$q'(t) = q(t) + \delta q(t)$$

$$t'(t) = t + \delta t(t)$$

+ Lagrangian will trans. so that it leaves the int. action

$$\delta \mathcal{S} = 0$$

But what are the consequences?

$$\mathcal{E}(q, \dot{q}, \ddot{q})\, \delta q + \frac{\mathrm{d} J(q, \dot{q})}{\mathrm{d}t} = 0$$

is a constant!

For time translation invariance:

$$\delta t = \epsilon, \quad \delta q = -\epsilon \dot{q}$$

$$E := \frac{\partial L}{\partial \dot{q}} \dot{q} - L$$

Conserved Energy function!

Closer to Noether's theorem - Nonlocality

But, very **important**! The proof involves integrations by parts...

$$\int_{t_0}^{t_1} \mathrm{d}t \left\{ \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(q,\dot{q})\delta t\right] \right\} \quad + \quad \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right] - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\delta q = \frac{\partial L}{\partial \dot{q}}\delta \dot{q}$$

But, could we apply this step when the Lagrangian is **nonlocal?**

$$L([q],t) = q(t)\,(G*q)_{(t)} = q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(t-\sigma)q(\sigma)$$

There are **no derivatives...**

$$= q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(-\sigma)q(\sigma+t) = \sum_{n=0}^{\infty} a_n\, q(t)q^{(n)}(t)$$

$$\left(a_n := \int_{\mathbb{R}} \mathrm{d}\sigma \frac{\sigma^n G(-\sigma)}{n!}\right)$$

$$q(\sigma+t) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} q^{(n)}(t)$$

But, we have to **integrate by parts infinite times...**

Noether's theorem for nonlocal Lagrangians

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Nonlocal Lagrangian fields: Noether's theorem and Hamiltonian formalism

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This article aims to study nonlocal Lagrangians with an infinite number of
We obtain an extension of Noether's theorem and Noether's identities for such Lagra
a Hamiltonian formalism for them. In addition, we show that n-order local Lagrangi
particular case, and the standard results can be recovered. Finally, this formalism is
p-adic open string field.

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I. INTRODUCTION

One of the most frequently emerging features in quantum gravity models is nonlocality. In string theory, for instance, nonlocality is displayed in its interactions, characterized by its infinite derivative structure [1,2]. In a more

mere curiosity presented
nonlocality is a very prob
initial value problem an
However, recent studies [
problem might be well p

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Non-local Lagrangian mechanics: Noether's theorem and Hamiltonian formalism

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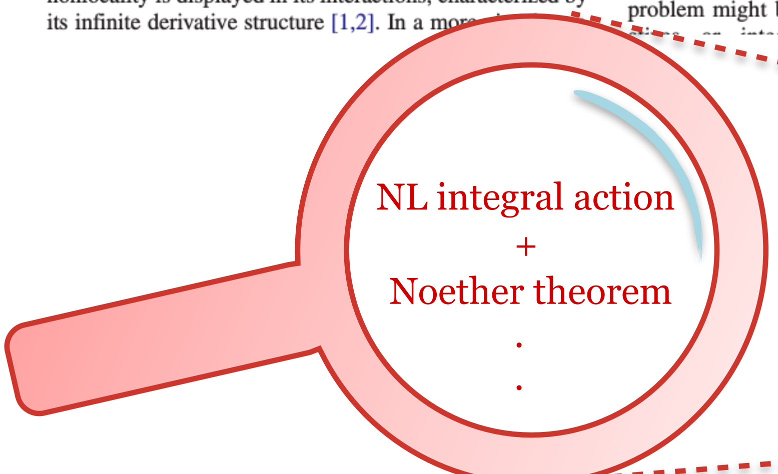
Abstract
Lagrangian systems with a finite number of degrees of freedom that are non-local in time are studied. We obtain an extension of Noether's theorem and Noether identities to this kind of Lagrangians. A Hamiltonian formalism has then been set up for these systems. n-order local Lagrangians can be treated as a particular case of non-local ones and standard results are recovered. The method is then applied to several other cases, namely two examples of non-local oscillators and the p-adic particle.



The 1-parameter family of finite action integrals

$$S(q,R) = \int_{|t|\leq R} \mathrm{d}t\, L(T_t q, t)\,, \qquad \forall R \in \mathbb{R}^+$$

Time evolution operator

Ex:

$$L(T_t q) = q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(t-\sigma) q(\sigma) = q(t)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(-\sigma) q(\sigma+t) = T_t q(0)\int_{\mathbb{R}} \mathrm{d}\sigma\, G(-\sigma) T_t q(\sigma)$$

Noether's Theorem for NL Lagrangians

- Consider the infinitesimal transformations: $\begin{cases} t'(t) = t + \delta t(t) \\ q'(t) = q(t) + \delta q(t) \end{cases}$

The NL Lagrangian will transform so that it leaves the NL action integral invariant, namely,

$$\int_{t_0'}^{t_1'} \mathrm{d}t' L'(T_{t'} q', t') = \int_{t_0}^{t_1} \mathrm{d}t\, L(T_t q, t) \quad \Rightarrow \quad \delta \mathcal{S} = 0$$

Noether's theorem

The procedure to prove it is exactly the same, except for one step! Instead of integrating by parts, we use a (more complex) "trick" to get the boundary terms!

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot q}\delta q\right] - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}\delta q = \frac{\partial L}{\partial \dot q}\delta \dot q$$

$$\lambda(q,t,t+\zeta)\delta q(t+\xi) - \lambda(q,t-\xi,t)\delta q(t) = \int_0^1 \mathrm{d}\eta \frac{\partial}{\partial \eta}\left[\lambda(q,t+(\eta-1)\xi,t+\eta\xi)\delta q(t+\eta\xi)\right] = \xi\int_0^1 \mathrm{d}\eta \frac{\partial}{\partial t}\left[\lambda(q,t+(\eta-1)\xi,t+\eta\xi)\delta q(t+\eta\xi)\right]$$

$$\lambda(q,t,\sigma) = \frac{\delta L(T_t q,t)}{\delta q(\sigma)}$$

$$U(T_t q,t) = \int_{\mathbb{R}} \mathrm{d}\rho\, \delta q(t+\rho) P(q,t,\rho)$$

$$P(q,t,\rho) = \int_{\mathbb{R}} \mathrm{d}\zeta\, [\theta(\rho)-\theta(\zeta)]\frac{\delta L(T_{t+\zeta}q,t+\zeta)}{\delta q(t+\rho)}$$

The result: $= 0$

$$\mathcal{E}([q],t)\delta q(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[L(T_t q,t)\delta t(t) + U(T_t q,t)\right] \equiv 0$$

$J(T_t q,t)$ is a constant!

Energy function - Nonlocal Physics

- Consider the time-translation transformation $\begin{cases} \delta t = \epsilon \\ \delta q = -\epsilon\, \dot{q} \end{cases}$

Def: The energy function is

$$E(T_t q) = -\epsilon^{-1} J(T_t q) = \int_{\mathbb{R}} \mathrm{d}\rho\, \dot{q}(\rho + t) P(q, \rho, t) - L(T_t q)$$

with

$$P(q, \rho, t) = \int_{\mathbb{R}} \mathrm{d}\zeta [\theta(\rho) - \theta(\zeta)] \frac{\delta L(T_{\zeta+t} q)}{\delta q(\rho + t)}$$

Check: $L(T_t q) := L(T_t \dot{q}_0, T_t q_0) = L(\dot{q}(t), q(t)) \Rightarrow E(q, \dot{q}) = \frac{\partial L(\dot{q}, q)}{\partial \dot{q}} \dot{q} - L(q, \dot{q})$!!

- Application to Nonlocal Physics: Dispersive Media

$$S(\tilde{A}, R) = \frac{1}{4} \int_{|x| \leq R} \mathrm{d}x\, \tilde{F}_{ab}(x) \left(M^{abcd} * \tilde{F}_{cd} \right)_{(x)}$$

Results for a wave package:

$$\mathcal{U} \approx \frac{1}{4} \mathbb{R}\mathrm{e} \left[\frac{\mathrm{d}(\varepsilon\omega)}{\mathrm{d}\omega} \underline{\tilde{\mathbf{E}}} \cdot \underline{\tilde{\mathbf{E}}}^* + \frac{\mu^*}{\mu} \frac{\mathrm{d}(\mu\omega)}{\mathrm{d}\omega} \underline{\tilde{\mathbf{H}}} \cdot \underline{\tilde{\mathbf{H}}}^* \right]$$

New...

$$G^i \approx \frac{1}{4} \mathbb{R}\mathrm{e} \left[\frac{1}{\omega\mu} \frac{\mathrm{d}(\varepsilon\mu\omega^2)}{\mathrm{d}\omega} \underline{\tilde{\mathbf{E}}} \times \underline{\tilde{\mathbf{B}}}^* \right]$$

Conclusions

- Noether's theorem connects the symmetries of our system with the conservation laws.

- Noether’s theorem involves theories with local operators (derivatives). What if it involves pseudo-differential operators or integro-differential operators? $\mathcal{O}(f)_{(x)} = (G * f)_{(x)}$

- We propose an extension of Noether’s theorem for them.

Conclusions

Observations:

- All of the above we have extended to classical field theory.
- Some applications:
  - Nonlocal electrodynamics - Dispersive Media.
  - Nonlocal harmonic oscillator.
  - p-adic strings.
  - Non-commutative theories (End of July).

Thank you!

Any question?