Extending Noether's Theorem to Nonlocal Theories

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Motivation

The exploration of global and local symmetries in the realm of convolutional neural network (CNN) architectures has emerged as an intriguing area of research. In [1] suggest that symmetries may hold significant roles in the construction of CNN architectures, and that understanding the associated conserved quantities can provide valuable insights into the dynamics of network evolution during training. In order to accomplish this objective, the first requirement is an expansion of Noether's theorem [2] that applies to nonlocal theories. Hence, this poster presents a novel proof of the extension of Noether's theorem to nonlocal Lagrangians characterized by an infinite number of degrees of freedom. By establishing this extension, we unlock new possibilities for understanding complex systems governed by nonlocal operators, such as, the convolution one. It relies on [3], and, up until now, this extension has demonstrated successful application across various scenarios. These include p-adic particles/strings [4,5], different examples of non-local harmonic oscillators [5], and electrodynamics of dispersive media [6]. In each of these cases, it has proven capable of replicating established results and, most importantly, has yielded novel results.

Extending Noether's Theorem: A Proof for Nonlocal Lagrangians

Consider the infinitesimal transformations

$$x'^a(x) = x^a + \delta x^a(x) \qquad \text{and} \qquad \tilde{\phi}'^A(x) = \tilde{\phi}^A(x) + \delta\tilde{\phi}^A(x)\,. \tag{1}$$

The nonlocal Lagrangian density $\mathcal{L}(T_x\tilde{\phi}^A, x)$ transforms so that the action integral over any four-volume is preserved $S'(\mathcal{V}') = S(\mathcal{V})$, namely,

$$\mathcal{L}'(T_{x'}\tilde{\phi}'^A, x') = \mathcal{L}(T_x\tilde{\phi}^A, x)\left|\frac{\partial x}{\partial x'}\right|\,, \tag{2}$$

where $T_x\tilde{\phi}^A(y) = \tilde{\phi}^A(y+x)$. Therefore, if $\mathcal{V}'$ is the transformation of the spacetime volume $\mathcal{V}$ according to (1), we get

$$\int_{\mathcal{V}'} \mathrm{d}x\, \mathcal{L}'(T_x\tilde{\phi}'^A, x) - \int_{\mathcal{V}} \mathrm{d}x\, \mathcal{L}(T_x\tilde{\phi}^A, x) = 0\,, \tag{3}$$

where we have replaced the dummy variable x' with x. Let us assume that the volumes $\mathcal{V}'$ and $\mathcal{V}$ share a large region and only differ in an infinitesimal layer close to the boundary $\partial\mathcal{V}$. If $\mathrm{d}\Sigma_a$ is the hypersurface element on the boundary, then the volume element close to the boundary is $\mathrm{d}x = \mathrm{d}\Sigma_b\,\delta x^b$. Hence, by neglecting second-order infinitesimals, equation (3) becomes

$$\int_{\mathcal{V}} \mathrm{d}x\, \left[\mathcal{L}'(T_x\tilde{\phi}'^A, x) - \mathcal{L}(T_x\tilde{\phi}^A, x)\right] + \int_{\partial\mathcal{V}} \mathcal{L}(T_x\tilde{\phi}^A, x)\,\delta x^b\,\mathrm{d}\Sigma_b = 0\,. \tag{4}$$

For a Noether symmetry, we have that

$$\mathcal{L}'(T_x\tilde{\phi}'^A, x) = \mathcal{L}(T_x\tilde{\phi}'^A, x) + \partial_b W^b(T_x\tilde{\phi}'^A, x)\,, \tag{5}$$

where $W^b(T_x\tilde{\phi}'^A, x)$ is a first-order infinitesimal; therefore,

$$\mathcal{L}'(T_x\tilde{\phi}'^A, x) - \mathcal{L}(T_x\tilde{\phi}^A, x) = \partial_b W^b(T_x\tilde{\phi}^A, x) + \int_{\mathbb{R}^4} \mathrm{d}y\,\lambda_A(\tilde{\phi}, x, y)\,\delta\tilde{\phi}^A(y) \qquad \text{where} \qquad \lambda_A(\tilde{\phi}, x, y) = \frac{\delta\mathcal{L}(T_x\tilde{\phi}^A, x)}{\delta\tilde{\phi}^A(y)}\,, \tag{6}$$

where ∂_b is the partial derivative for x^b, and second-order infinitesimals have been neglected. Introducing the variable $z = y - x$ in (6), substituting it in (4), and applying Gauss' theorem, we obtain that

$$\int_{\mathcal{V}} \mathrm{d}x\, \left\{\partial_b\left[\mathcal{L}(T_x\tilde{\phi}^A, x)\,\delta x^b + W^b(T_x\tilde{\phi}^A, x)\right] + \int_{\mathbb{R}^4} \mathrm{d}z\,\lambda_A(\tilde{\phi}, x, z+x)\,\delta\tilde{\phi}^A(z+x)\right\} = 0\,. \tag{7}$$

Furthermore, including the nonlocal Euler-Lagrange equations

$$\psi_A(\tilde{\phi}, x) = \int_{\mathbb{R}^4} \mathrm{d}y\,\lambda_A(\tilde{\phi}, y, x)\,, \tag{8}$$

we can write

$$-\int_{\mathcal{V}} \mathrm{d}x\,\psi_A(\tilde{\phi}, x)\,\delta\tilde{\phi}^A(x) = \int_{\mathcal{V}} \mathrm{d}x\, \left\{\partial_b\left[\mathcal{L}(T_x\tilde{\phi}^A, x)\,\delta x^b + W^b(T_x\tilde{\phi}^A, x)\right] + \int_{\mathbb{R}^4} \mathrm{d}z\,\left[\lambda_A(\tilde{\phi}, x, z+x)\,\delta\tilde{\phi}^A(z+x) - \lambda_A(\tilde{\phi}, x-z, x)\,\delta\tilde{\phi}^A(x)\right]\right\}\,. \tag{9}$$

Now, the trick. We use the identity

$$\begin{aligned}\lambda_A(\tilde{\phi}, x, z+x)\,\delta\tilde{\phi}^A(z+x) - \lambda_A(\tilde{\phi}, x-z, x)\,\delta\tilde{\phi}^A(x) &= \int_0^1 \mathrm{d}s\,\frac{\mathrm{d}}{\mathrm{d}s}\left\{\lambda_A(\tilde{\phi}, x+[s-1]z, x+sz)\,\delta\tilde{\phi}^A(x+sz)\right\}\\ &= \int_0^1 \mathrm{d}s\, z^b\,\frac{\partial}{\partial x^b}\left\{\lambda_A(\tilde{\phi}, x+[s-1]z, x+sz)\,\delta\tilde{\phi}^A(x+sz)\right\}\end{aligned} \tag{10}$$

that, combined with (9), leads to

$$\int_{\mathcal{V}} \mathrm{d}x\,\left\{\psi_A(\tilde{\phi}, x)\,\delta\tilde{\phi}^A(x) + \frac{\partial}{\partial x^b}\left[\mathcal{L}(T_x\tilde{\phi}, x)\,\delta x^b + W^b(T_x\tilde{\phi}, x) + \Pi^b(T_x\tilde{\phi}, x)\right]\right\} = 0\,, \tag{11}$$

where $\Pi^b(T_x\tilde{\phi}, x)$ is

$$\Pi^b(T_x\tilde{\phi}, x) := \int_{\mathbb{R}^4} \mathrm{d}z\, z^b \int_0^1 \mathrm{d}s\,\lambda_A(\tilde{\phi}, x+[s-1]z, x+sz)\,\delta\tilde{\phi}^A(x+sz)\,. \tag{12}$$

As equation (11) holds for any spacetime volume $\mathcal{V}$, it follows that

$$N(\tilde{\phi}, x) := \partial_b J^b(T_x\tilde{\phi}, x) + \psi_A(\tilde{\phi}, x)\,\delta\tilde{\phi}^A(x) \equiv 0\,, \tag{13}$$

where

$$J^b(T_x\tilde{\phi}, x) : = \mathcal{L}(T_x\tilde{\phi}, x)\,\delta x^b + W^b(T_x\tilde{\phi}, x) + \int_{\mathbb{R}^4} \mathrm{d}z\, z^b \int_0^1 \mathrm{d}s\,\lambda_A(\tilde{\phi}, x+[s-1]z, x+sz)\,\delta\tilde{\phi}^A(x+sz)\,. \tag{14}$$

Equation (13) is an identity and holds for any kinematic field $\tilde{\phi}$. For dynamic fields, this identity implies that the current $J^b(T_x\tilde{\phi}, x)$ is locally conserved

$$\partial_b J^b = 0\,. \tag{15}$$

The angular momentum and energy-momentum currents

Let us particularise the conserved current (14) for a Poincaré symmetry. By substituting $\delta x^a = \varepsilon^a + \omega^a_{\ b} x^b$ and $\omega_{ab} + \omega_{ba} = 0$, where ε^a and $\omega^a_{\ b}$ are constants, $\omega_{ab} = \eta_{ac}\omega^c_{\ b}$ and $\eta_{ac} = \mathrm{diag}(1,1,1,-1)$ is the Minkowski matrix to raise and lower indices into (14) and assuming that the nonlocal Lagrangian density is Poincaré invariant — therefore, $W^b = 0$—, we find that the conserved current can be written as

$$J^b(T_x\tilde{\phi}, x) = -\varepsilon^a\,\mathcal{T}_a^{\ b}(T_x\tilde{\phi}, x) - \frac{1}{2}\omega^{ac}\mathcal{J}_{ac}^{\ \ b}(T_x\tilde{\phi}, x)\,, \tag{16}$$

where

$$\mathcal{T}_a^{\ b} := -\mathcal{L}(T_x\tilde{\phi}, x)\,\delta_a^b + \int_{\mathbb{R}^4} \mathrm{d}z\, z^b \int_0^1 \mathrm{d}s\,\lambda_A(\tilde{\phi}, x+[s-1]z, x+sz)\,\tilde{\phi}^A_{|a}(x+sz)\,, \tag{17}$$

and $\mathcal{J}_{ac}^{\ \ b} := 2\,x_{[c}\mathcal{T}_{a]}^{\ \ b} + \mathcal{S}_{ac}^{\ \ b}$ with

$$\mathcal{S}_{ac}^{\ \ b}(T_x\tilde{\phi}, x) := 2\int_{\mathbb{R}^4} \mathrm{d}z\, z^b \int_0^1 \mathrm{d}s\,\lambda_A(\tilde{\phi}, x+[s-1]z, x+sz)\left[s\, z_{[c}\tilde{\phi}^A_{|a]}(x+sz) - M^A_{B[ac]}\tilde{\phi}^B(x+sz)\right] \tag{18}$$

are the canonical energy-momentum tensor, the angular momentum tensor, the orbital angular momentum tensor, and the spin current, respectively. Since the ten parameters ε^a and ω^{ac} are independent, the local conservation of the current $J^b(T_x\tilde{\phi}, x)$ implies that the currents $\mathcal{T}_a^{\ b}(T_x\tilde{\phi}, x)$ and $\mathcal{J}_{ac}^{\ \ b}(T_x\tilde{\phi}, x)$ are separately conserved, that is,

$$\partial_b\mathcal{T}_a^{\ b}(T_x\tilde{\phi}, x) = 0 \qquad \text{and} \qquad \partial_b\mathcal{J}_{ac}^{\ \ b}(T_x\tilde{\phi}, x) = 0\,, \qquad \text{or} \qquad \partial_b\mathcal{T}_a^{\ b} = 0 \qquad \text{and} \qquad \partial_b\mathcal{S}_{ac}^{\ \ b} + 2\,\mathcal{T}_{[ac]} = 0\,. \tag{19}$$

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