

Decoding Non-Locality: Lagrangian and Hamiltonian formulations, and Noether's theorem

Center for Theoretical Physics of the Universe, IBS Korea

C. Heredia Pimienta

Universitat de Barcelona, Departament de Física Quàntica i Astrofísica
Institute of Cosmos Sciences (ICCUB)



Nonlocal Lagrangian: **Mechanics**

- Definitions: What does
 - the extended kinematic space
 - a nonlocal Lagrangian
 - temporal evolution mean?
- Principle of least action
 - Nonlocal Euler-Lagrange equations
 - Two ways of coordinating the extended kinematic space
 - The extended dynamic space
- **Noether's theorem:** Energy function
- Hamiltonian formalism: Legendre transformation

Motivations

- Example in physics with nonlocality:

Classical electrodynamics in dispersive media

$$S(\tilde{A}, R) = \frac{1}{4} \int_{|x| \leq R} dx \tilde{F}_{ab}(x) \left(M^{abcd} * \tilde{F}_{cd} \right) (x)$$

$$M^{abcd}(x) = (2\pi)^{-2} \left[m(x) \hat{\eta}^{a[c} \hat{\eta}^{d]b} + 2\epsilon(x) u^{[a} \hat{\eta}^{b][c} u^{d]} \right]$$

That is why we propose this structure...

Motivated by: $\epsilon_0, m_0 \longrightarrow \epsilon(\mathbf{k}, \omega), m(\mathbf{k}, \omega)$

$$\left. \begin{array}{l} \mathbf{D}(\mathbf{k}, \omega) \\ \mathbf{H}(\mathbf{k}, \omega) \end{array} \right\} \begin{array}{l} \mathbf{D}(\mathbf{k}, \omega) = \epsilon(\mathbf{k}, \omega) \mathbf{E}(\mathbf{k}, \omega) \\ \mathbf{H}(\mathbf{k}, \omega) = m(\mathbf{k}, \omega) \mathbf{B}(\mathbf{k}, \omega) \end{array} \longrightarrow \begin{array}{l} \mathbf{D}(x) = (\epsilon * \mathbf{E})_{(x)} \\ \mathbf{H}(x) = (m * \mathbf{B})_{(x)} \end{array}$$

- Consider the following Lagrangian:

$$L([q], t) = q(t) (G * q)_{(t)} = q(t) \int_{\mathbb{R}} d\sigma G(t - \sigma) q(\sigma)$$

How do we get the EOMs or the momenta?

$$= q(t) \int_{\mathbb{R}} d\sigma G(-\sigma) q(\sigma + t)$$

and then apply Taylor... $q(\sigma + t) = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} q^{(n)}(t)$

$$= \sum_{n=0}^{\infty} a_n q(t) q^{(n)}(t) \quad \left(a_n := \int_{\mathbb{R}} d\sigma \frac{\sigma^n G(-\sigma)}{n!} \right)$$

But, I have to integrate by parts infinite times...

Let us write it as

$$\begin{aligned} L([q], t) &= q(t) (G * q)_{(t)} = \sum_{n=0}^{\infty} a_n q(t) q^{(n)}(t) \\ &= \lim_{N \rightarrow \infty} \sum_{n=0}^N q(t) q^{(n)}(t) \end{aligned}$$

then we apply the Ostrogradski formalism to get:

EOMs: $\lim_{N \rightarrow \infty} \sum_{n=0}^N \left(-\frac{d}{dt} \right)^n \left[\frac{\partial L}{\partial q^{(n)}} \right] = 0$ Differential equations and moments are no longer correctly defined.

Momenta: $p_n := \lim_{N \rightarrow \infty} \sum_{k=n+1}^N \left(-\frac{d}{dt} \right)^{k-n-1} \frac{\partial L}{\partial q^{(k)}}$

Definitions

- **Def:** The class of all (possible) kinematic trajectories is the

$$\textit{kinematic space} \quad \mathcal{K} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^s)$$

$$q^i(t) \quad (i = 1, \dots, s)$$

- **Def:** The *extended kinematic space* is $\mathcal{K}' = \mathcal{K} \times \mathbb{R}$

for time-dependent Lagrangians

- **Def:** A *nonlocal Lagrangian* is a real-valued functional on \mathcal{K}'

$$([q^i], t) \in \mathcal{K}' \longrightarrow L([q^i], t) \in \mathbb{R}$$

where $[q^i]$ means functional dependence on the whole $q^i(\sigma)$

$$\mathbf{Ex:} \quad L([q], t) = q(t) (G * q)_{(t)} = q(t) \int_{\mathbb{R}} d\sigma G(t - \sigma) \underline{q(\sigma)}$$

- **Def:** We define the *time evolution operator* T_t as

$$([q^i], 0) \xrightarrow{T_t} ([T_t q^i], t) \quad \text{where} \quad T_t q^i(\sigma) = q^i(\sigma + t)$$

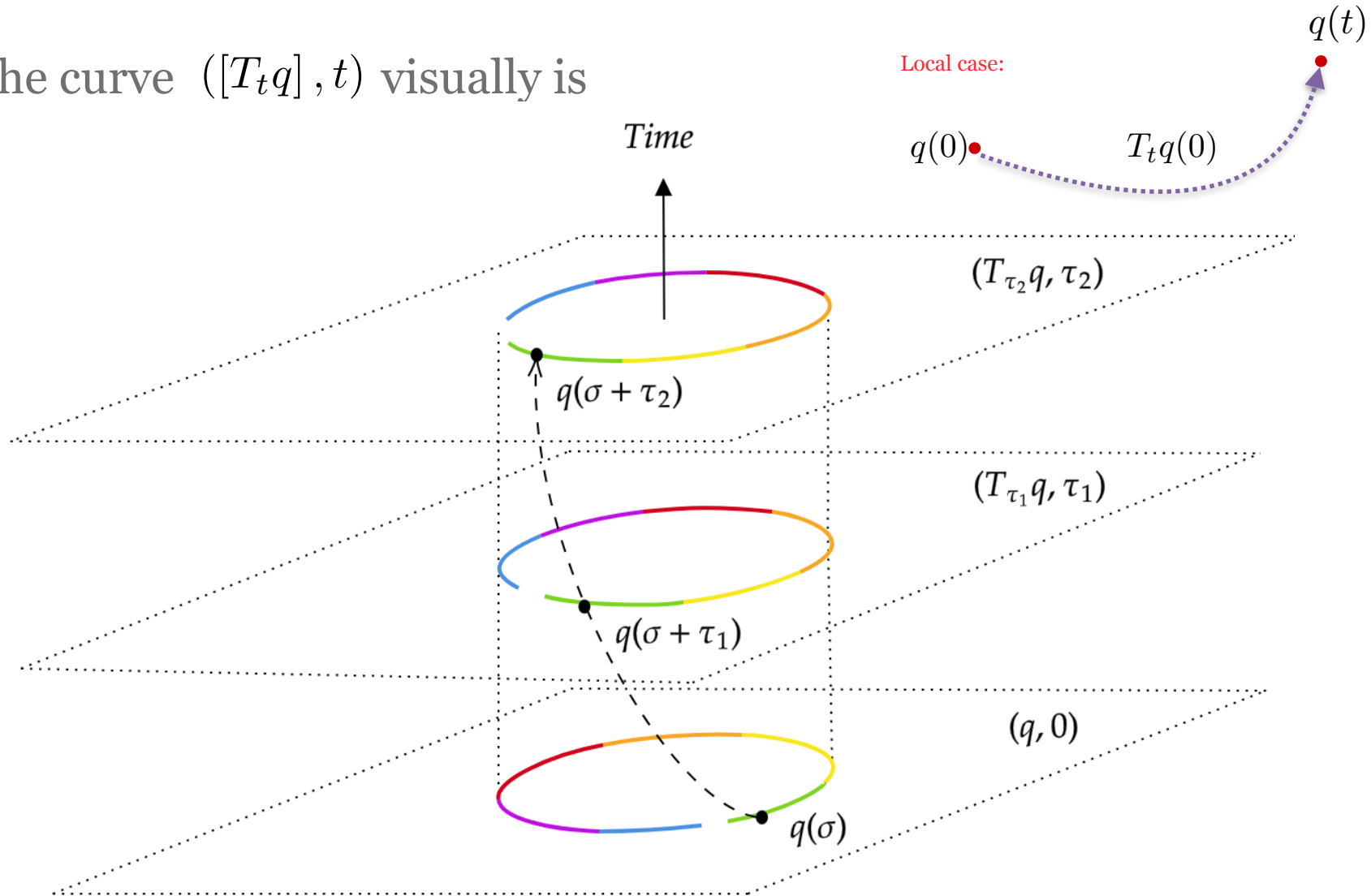
$$\begin{aligned} \mathbf{Ex:} \quad L(T_t q) &= q(t) \int_{\mathbb{R}} d\sigma G(t - \sigma) q(\sigma) = q(t) \int_{\mathbb{R}} d\sigma G(-\sigma) q(\sigma + t) \\ &= T_t q(0) \int_{\mathbb{R}} d\sigma G(-\sigma) (T_t q(\sigma)) \end{aligned}$$

- **Obs:** It is possible to establish a one-to-one correspondence between the “infinite-order” Ostrogradsky formalism and the nonlocal one via *formal Taylor’s series (FTS)*

$$\left(\left\{ q^{i,(r)}(t) \right\}_{r \in \mathbb{N}}, t \right) \longleftrightarrow ([q^i], t) \quad \text{with} \quad q^i(\sigma + t) = \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} q^{i,(k)}(t)$$

Definitions

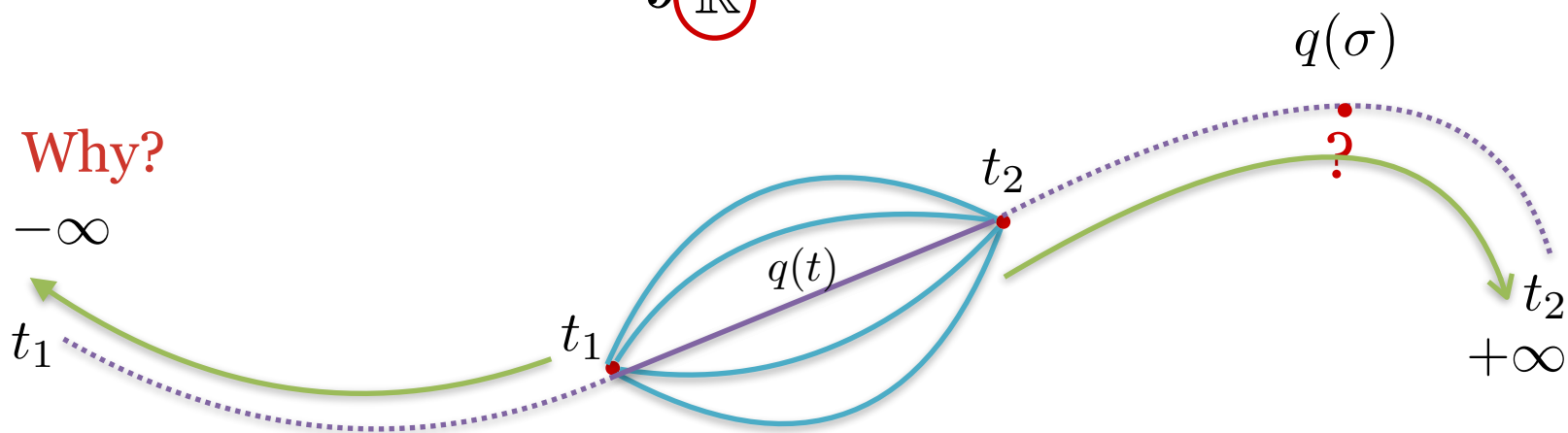
The curve $([T_t q], t)$ visually is



Principle of least action

- **Def:** We define the *nonlocal action integral* as

$$S(q) := \int_{\mathbb{R}} dt L(T_t q, t)$$



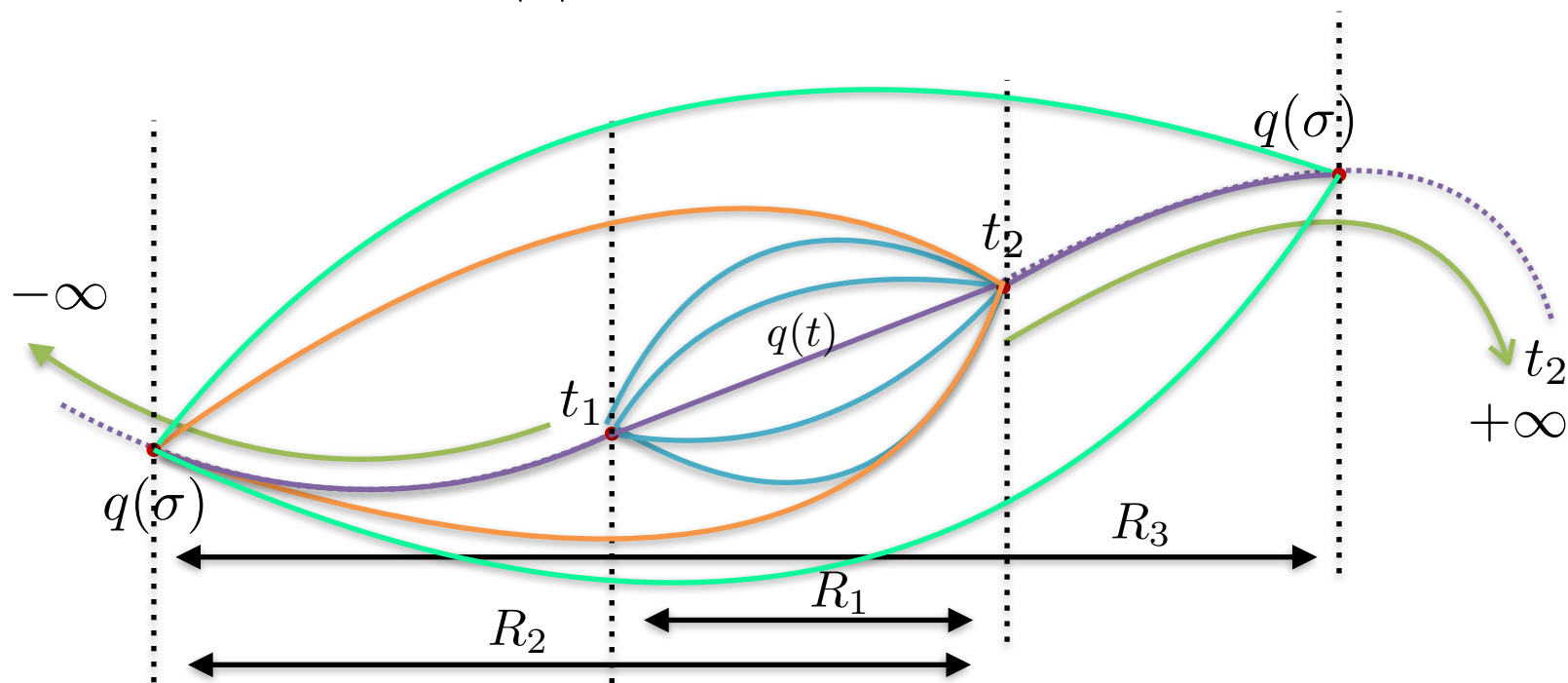
where $t \in [t_1, t_2]$ but, in nonlocal theories, $q(\sigma)$ with $\sigma \in \mathbb{R}$

If it is unbounded, could it be infinity?

It could be. For this reason, we introduce:

the 1-parameter family of finite action integrals

$$S(q, R) = \int_{|t| \leq R} dt L(T_t q, t), \quad \forall R \in \mathbb{R}^+$$



- The principle of least action reads

$$\begin{aligned} \lim_{R \rightarrow \infty} \delta S(q, R) &\equiv \lim_{R \rightarrow \infty} \int_{|t| \leq R} dt \int_{\mathbb{R}} d\sigma \frac{\delta L(T_t q, t)}{\delta q(\sigma)} \delta q(\sigma) \\ &= \int_{\mathbb{R}} d\sigma \delta q(\sigma) \int_{\mathbb{R}} dt \frac{\delta L(T_t q, t)}{\delta q(\sigma)} = 0 \end{aligned}$$

$\forall \delta q(\sigma)$ of compact support.

It can be shown that, with a standard Lagrangian of the first order, we can recover the Euler-Lagrange equations

- As long as the limit exists, the **nonlocal Euler-Lagrange equations**

$$\forall \sigma \in \mathbb{R}$$

Cauchy's problem!

$$\lambda(q, t, \sigma) := \frac{\delta L(T_t q, t)}{\delta q(\sigma)}$$

$$\psi(q, \sigma) := \int_{\mathbb{R}} dt \lambda(q, t, \sigma)$$

are:

$$\psi(q, \sigma) = 0$$

- We might coordinate a point $z \in \mathcal{K}'$ in two different ways:

$$\begin{array}{l} \text{(a)} \quad \mathcal{K}' \longrightarrow \mathcal{K} \times \mathbb{R} \\ \quad \quad z \longmapsto (\tilde{q}, t) \end{array}$$

$$\begin{array}{l} \text{(b)} \quad \mathcal{K}' \longrightarrow \mathcal{K} \times \mathbb{R} \\ \quad \quad z \longmapsto (q, t) \end{array}$$

where $q(\sigma) = T_t \tilde{q}(\sigma) = \tilde{q}(\sigma + t)$

We shall refer to:

(\tilde{q}, t) as **static coordinates**

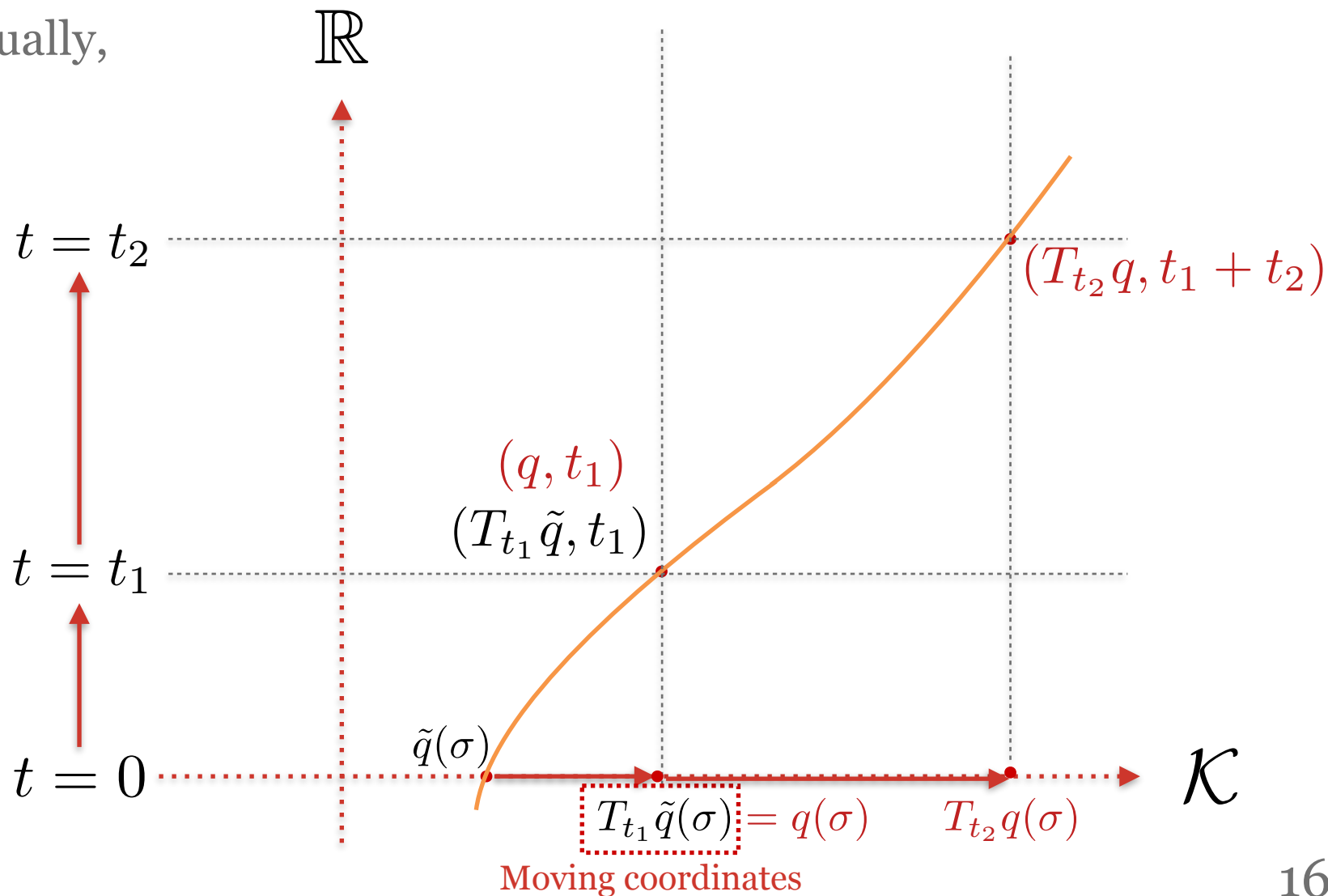
(q, t) as **moving coordinates**

- **Why?** Because the time evolution expression in each of these coordinate systems is:

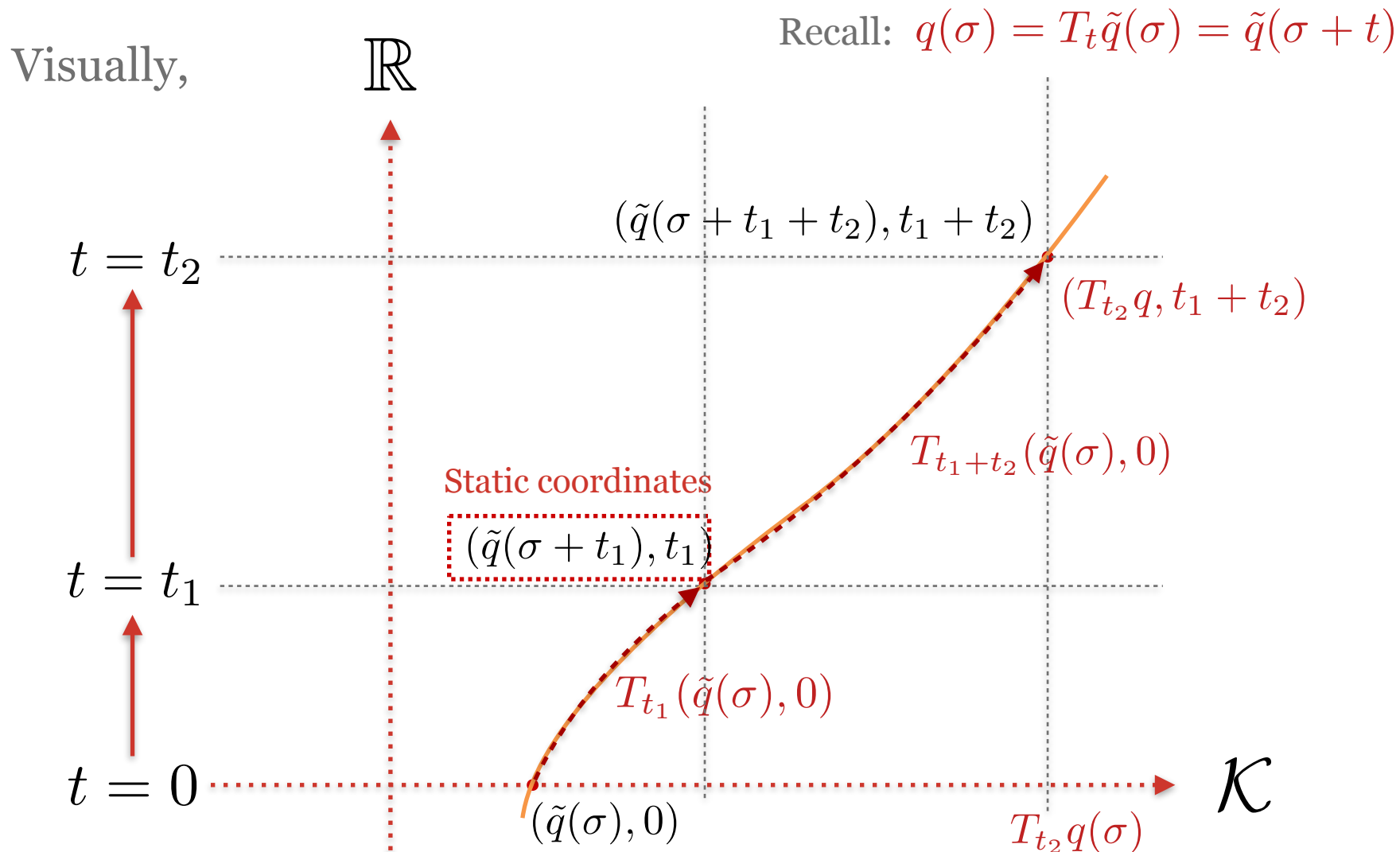
$$\text{(a)} \quad (\tilde{q}, t) \xrightarrow{T_\tau} (\tilde{q}, t + \tau) \quad \text{(b)} \quad (q, t) \xrightarrow{T_\tau} (T_\tau q, t + \tau)$$

Principle of least action

Visually,



Principle of least action



- Therefore, going back again to the nonlocal Euler-Lagrange equations,

$$\psi(q, \sigma) = 0 \quad \text{where} \quad \psi(q, \sigma) := \int_{\mathbb{R}} dt \lambda(q, t, \sigma)$$
$$\lambda(q, t, \sigma) := \frac{\delta L(T_t q, t)}{\delta q(\sigma)}$$

From slide 10:

$$L(T_t q) = q(t) \int_{\mathbb{R}} d\sigma G(-\sigma) q(\sigma + t)$$

We see that they are obtained in the context of *static coordinantes*, limited to trajectories

$$(T_t q, t) \in \mathcal{K}' \quad \text{starting at} \quad (q, 0)$$

- Therefore, they should be better understood:

$$\psi(\tilde{q}, \sigma) = 0 \quad \text{with} \quad \psi(\tilde{q}, \sigma) := \int_{\mathbb{R}} dt \lambda(\tilde{q}, t, \sigma)$$

$$\lambda(\tilde{q}, t, \sigma) := \frac{\delta L(T_t \tilde{q}, t)}{\delta \tilde{q}(\sigma)}$$

- In moving coordinates, but **why? I want my trajectory to start at**

(\tilde{q}, t) and not at $(\tilde{q}, 0)$

$$L(q, t) \sim e^{t} f(q)$$

$$L(T_t \tilde{q}, t) \xrightarrow[\substack{q = T_t \tilde{q} \\ (T_\tau q, t + \tau)}]{\quad} L(T_\tau q, t + \tau)$$

$$\Psi(q, t, \sigma) = 0 \quad \text{with} \quad \Psi(q, t, \sigma) \equiv \psi(\tilde{q}, t + \sigma)$$

Notice that: $\psi(\tilde{q}, \sigma) = \Psi(q, 0, \sigma)$

- The **infinitesimal time generator** \mathbf{D} acts on them

$$\mathbf{D}\psi(\tilde{q}, \sigma) = \partial_t \psi(\tilde{q}, \sigma)$$

$$\mathbf{D}\Psi(q, t, \sigma) = \left[\frac{\partial \Psi(T_\epsilon q, t + \epsilon, \sigma)}{\partial \epsilon} \right]_{\epsilon=0}$$

- Indeed, the nonlocal Euler-Lagrange equations are stable under time evolution:

$$\mathbf{D}\psi(\tilde{q}, \sigma) = 0 \quad \text{or} \quad \mathbf{D}\Psi(q, t, \sigma) = 0$$

- **Def:** The **extended dynamic space** \mathcal{D}' is the class of all dynamic trajectories, namely, those (q, t) that satisfy the nonlocal Euler-Lagrange equations. So, $\mathcal{D}' \subset \mathcal{K}'$
- **Obs:** As a rule, there are **no general theorems of existence and uniqueness** for integrodifferential equations.

Therefore, coordinating the dynamic space with the initial conditions will depend on the case.

- An alternative view of the nonlocal Euler-Lagrange equations:

They are the constraints (in implicit form) that define the extended dynamic space as a submanifold of the extended kinematic space.

Noether's theorem

Noether's Theorem for nonlocal Lagrangians

- Consider the infinitesimal transformations:

$$t'(t) = t + \delta t(t) \quad \text{and} \quad \tilde{q}'(t) = \tilde{q}(t) + \delta \tilde{q}(t)$$

The nonlocal Lagrangian will transform so that it leaves the nonlocal action integral invariant, namely,

$$L'(T_{t'} \tilde{q}', t') = \left| \frac{dt}{dt'} \right| L(T_t \tilde{q}, t)$$

Thus, if $[t_0, t_1]$ is a time interval and $[t'_0, t'_1]$ is the transformed one,

$$\int_{t'_0}^{t'_1} dt' L'(T_{t'} \tilde{q}', t') = \int_{t_0}^{t_1} dt L(T_t \tilde{q}, t)$$

- As $S'(\tilde{q}', t'_0, t'_1) = S(\tilde{q}, t_0, t_1)$:

$$\int_{t'_0}^{t'_1} dt L'(T_t \tilde{q}', t) - \int_{t_0}^{t_1} dt L(T_t \tilde{q}, t) = 0$$

where the dummy variable t' is replaced with t

- Given those infinitesimal transformations, the first integral can be approximated to the leading order as

$$\int_{t'_0}^{t'_1} dt L'(T_t \tilde{q}', t) = \int_{t_0}^{t_1} dt \left\{ L'(T_t \tilde{q}', t) + \frac{d}{dt} [L(T_t \tilde{q}, t) \delta t] \right\}$$

Therefore,

$$\int_{t_0}^{t_1} dt \left\{ L'(T_t \tilde{q}', t) - L(T_t \tilde{q}, t) + \frac{d}{dt} [L(T_t \tilde{q}, t) \delta t] \right\} = 0$$

- We say that a transformation is a Noether symmetry if

$$L'(T_t \tilde{q}', t) = L(T_t \tilde{q}', t) + \frac{d}{dt} W(T_t \tilde{q}', t)$$

- These quantities become up to the leading order:

$$L(T_t \tilde{q}', t) \stackrel{q'(t) = q(t) + \delta q(t)}{\approx} L(T_t \tilde{q}, t) + \int_{\mathbb{R}} d\sigma \frac{\delta L(T_t \tilde{q}, t)}{\delta \tilde{q}(\sigma)} \delta \tilde{q}(\sigma) \lambda(\tilde{q}, t, \sigma)$$

$$W(T_t \tilde{q}', t) \approx W(T_t \tilde{q}, t) + \int_{\mathbb{R}} d\sigma \frac{\delta W(T_t \tilde{q}, t)}{\delta \tilde{q}(\sigma)} \delta \tilde{q}(\sigma)$$

Therefore,

$$L'(T_t \tilde{q}', t) = L(T_t \tilde{q}, t) + \frac{d}{dt} W(T_t \tilde{q}, t) + \int_{\mathbb{R}} d\sigma \lambda(\tilde{q}, t, \sigma) \delta \tilde{q}(\sigma)$$

- Let us introduce the last equation into

$$\int_{t_0}^{t_1} dt \left\{ L'(T_t \tilde{q}', t) - L(T_t \tilde{q}, t) + \frac{d}{dt} [L(T_t \tilde{q}, t) \delta t] \right\} = 0$$

- Thus,

$$\int_{t_1}^{t_2} dt \left\{ \delta L + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} + \frac{d}{dt} [L(q, \dot{q}, t) \delta t] \right\} \quad \text{Local case...}$$

$$\int_{t_0}^{t_1} dt \left\{ \int_{\mathbb{R}} d\sigma \lambda(\tilde{q}, t, \sigma) \delta \tilde{q}(\sigma) + \frac{d}{dt} [L(T_t \tilde{q}, t) \delta t + W(T_t \tilde{q}, t)] \right\} = 0$$

Crucial point! How do you get the boundary terms and EOMs?

$$\text{Recall: } \psi(\tilde{q}, \sigma) = 0 \quad \text{with} \quad \psi(\tilde{q}, \sigma) = \int_{\mathbb{R}} dt \lambda(\tilde{q}, t, \sigma)$$

- Let us add and subtract $\int_{\mathbb{R}} d\sigma \lambda(\tilde{q}, \sigma, t) \delta \tilde{q}(t) = \psi(\tilde{q}, t) \delta \tilde{q}(t)$

- We obtain:

$$\int_{t_0}^{t_1} dt \left\{ \psi(\tilde{q}, t) \delta\tilde{q}(t) + \frac{d}{dt} [L(T_t \tilde{q}, t) \delta t + W(T_t \tilde{q}, t)] \right. \\ \left. + \int_{\mathbb{R}} d\sigma \lambda(\tilde{q}, t, \sigma) \delta\tilde{q}(\sigma) - \int_{\mathbb{R}} d\sigma \lambda(\tilde{q}, \sigma, t) \delta\tilde{q}(t) \right\} = 0$$

Is there a way to link them?

- After a suitable change of variable:

$$\int_{\mathbb{R}} d\xi [\lambda(\tilde{q}, t, t + \xi) \delta\tilde{q}(t + \xi) - \lambda(\tilde{q}, t - \xi, t) \delta\tilde{q}(t)]$$

- Now, by using the following identity:

$$\begin{aligned} & \lambda(\tilde{q}, t, t + \xi) \delta\tilde{q}(t + \xi) - \lambda(\tilde{q}, t - \xi, t) \delta\tilde{q}(t) = \\ & = \int_0^1 d\eta \frac{\partial}{\partial \eta} [\lambda(\tilde{q}, t + (\eta - 1)\xi, t + \eta\xi) \delta\tilde{q}(t + \eta\xi)] \\ & = \xi \int_0^1 d\eta \frac{\partial}{\partial t} [\lambda(\tilde{q}, t + (\eta - 1)\xi, t + \eta\xi) \delta\tilde{q}(t + \eta\xi)] \end{aligned}$$

- We find:

$$\int_{t_0}^{t_1} dt \left\{ \psi(\tilde{q}, t) \delta \tilde{q}(t) + \frac{d}{dt} [L(T_t \tilde{q}, t) \delta t + W(T_t \tilde{q}, t) + U(T_t \tilde{q}, t)] \right\} = 0$$

where

$$U(T_t \tilde{q}, t) := \int_{\mathbb{R}} d\xi \xi \int_0^1 d\eta \lambda(\tilde{q}, t + (\eta - 1)\xi, t + \eta\xi) \delta \tilde{q}(t + \eta\xi)$$

• (After a bit of algebra...)

Using the analogy with the local case, this will be the Legendre transformation for nonlocal Lagrangians!

$$U(T_t \tilde{q}, t) = \int_{\mathbb{R}} d\rho \delta \tilde{q}(t + \rho) P(\tilde{q}, t, \rho) \quad \text{with}$$

$$P(\tilde{q}, t, \rho) := \int_{\mathbb{R}} d\zeta [\theta(\rho) - \theta(\zeta)] \frac{\delta L(T_{t+\zeta} \tilde{q}, t + \zeta)}{\delta \tilde{q}(t + \rho)}$$

Since the choice of the interval is completely arbitrary, we obtain:

- Noether's identity:

$$\psi(\tilde{q}, t) \delta\tilde{q}(t) + \frac{d}{dt} [L(T_t\tilde{q}, t) \delta t(t) + W(T_t\tilde{q}, t) + U(T_t\tilde{q}, t)] \equiv 0$$

- and Noether's conserved quantity: Of course: $\left. \frac{d}{dt} J(T_t\tilde{q}, t) \right|_{\psi=0} = 0$

$$J(T_t\tilde{q}, t) := L(T_t\tilde{q}, t) \delta t(t) + W(T_t\tilde{q}, t) + U(T_t\tilde{q}, t)$$

We have just extended Noether's theorem for nonlocal Lagrangians.

Energy function

- First of all, let us rewrite the Noether current in moving coordinates $q = T_t \tilde{q}$

Then,

$$J(q, t) = L(q, t) \delta t(t) + W(q, t) + U(q, t)$$

where

$$U(q, t) = \int_{\mathbb{R}} d\rho \delta q(\rho) P(q, t, \rho)$$

with

$$P(q, t, \rho) = \int_{\mathbb{R}} d\zeta [\theta(\rho) - \theta(\zeta)] \frac{\delta L(T_\zeta q, t + \zeta)}{\delta q(\rho)}$$

- Consider the **time-translation transformation**

$$\delta t = \epsilon \quad \text{and} \quad \delta q = -\epsilon \dot{q}$$

- **Def:** The **energy function** -in moving coordinates- is

$$E(q) := -\epsilon^{-1} J(q) = -L(q) + \int_{\mathbb{R}} d\rho \dot{q}(\rho) P(q, \rho)$$

with

$$P(q, \rho) := \int_{\mathbb{R}} d\zeta [\theta(\rho) - \theta(\zeta)] \frac{\delta L(T_{\zeta} q)}{\delta q(\rho)}$$

where we have assumed that the nonlocal Lagrangian does not explicitly depend on time $W(q) = 0$.

Considering a first-order Lagrangian, one can recover the usual structure of the energy function.

$$E(q_0, \dot{q}_0) := \frac{\partial L_L(q_0, \dot{q}_0)}{\partial \dot{q}_0} \dot{q}_0 - L_L(q_0, \dot{q}_0)$$

Hamiltonian formalism

- **Def:** The **nonlocal Hamiltonian** on the extended phase space $\Gamma' = \Gamma \times \mathbb{R}$ is defined -in moving coordinates- as follows:

$$H(q, \pi, t) = \int_{\mathbb{R}} d\sigma \pi(\sigma) \dot{q}(\sigma) - L(q, t)$$

that is equipped with the following **Poisson bracket**

$$\{F, G\} = \int_{\mathbb{R}} d\sigma \left(\frac{\delta F}{\delta q(\sigma)} \frac{\delta G}{\delta \pi(\sigma)} - \frac{\delta F}{\delta \pi(\sigma)} \frac{\delta G}{\delta q(\sigma)} \right)$$

- **Hamilton's equations** are $\left\{ \begin{array}{l} \mathbf{X}_H q(\sigma) = \dot{q}(\sigma) \\ \mathbf{X}_H \pi(\sigma) = \dot{\pi}(\sigma) + \frac{\delta L(q, t)}{\delta q(\sigma)} \end{array} \right.$ 34

where \mathbf{X}_H is the **Hamiltonian vector field**

$$\mathbf{X}_H = \partial_t + \int_{\mathbb{R}} d\sigma \left(\dot{q}(\sigma) \frac{\delta}{\delta q(\sigma)} + \left[\dot{\pi}(\sigma) + \frac{\delta L(q, t)}{\delta q(\sigma)} \right] \frac{\delta}{\delta \pi(\sigma)} \right)$$

- Hamilton's equations can be written in a more compact form by using the contact differential 2-form:

$$\Omega' = \Omega - \delta H \wedge \delta t \quad \text{with} \quad \Omega = \int_{\mathbb{R}} d\sigma \delta\pi(\sigma) \wedge \delta q(\sigma)$$

(differential)

Namely,

$$i_{\mathbf{X}_H} \Omega' = 0$$

So far, this Hamiltonian system has nothing to do with the Lagrangian formulation previously presented.

How can we connect them?

- We can connect both through the **injection**:

$$(q, t) \in \mathcal{D}' \xrightarrow{j} (q, \pi, t) \in \Gamma'$$

where

It is the prefactor in the Noether conserved quantity

$$\pi(\sigma) := P(q, t, \sigma) \quad \delta q(\sigma) P(q, t, \sigma)$$

- Def:** j defines a one-to-one map from \mathcal{D}' into $j(\mathcal{D}') \subset \Gamma'$ namely, a submanifold implicitly defined by the constraints:

$$\Psi(q, t, \sigma) = 0 \quad \text{and} \quad \Upsilon(q, \pi, t, \sigma) := \pi(\sigma) - P(q, t, \sigma) = 0$$

- Besides, $j^T \mathbf{D} = X_{\mathbf{H}}$

Consequently, the constraints are stable by the Hamiltonian flow.

- To translate the Hamiltonian formalism in Γ' into \mathcal{D}' , we use the **pullback** j^*

Therefore, the contact form $\omega' \in \Lambda^2(\mathcal{D}')$ is $\omega' := j^* \Omega'$

$$\omega'(q, t) = \omega(q, t) - \delta h(q, t) \wedge \delta t$$

where

$$\omega(q, t) = \int_{\mathbb{R}} d\sigma \delta P(q, t, \sigma) \wedge \delta q(\sigma)$$

It's close but it is not clear whether it is non-degenerate!

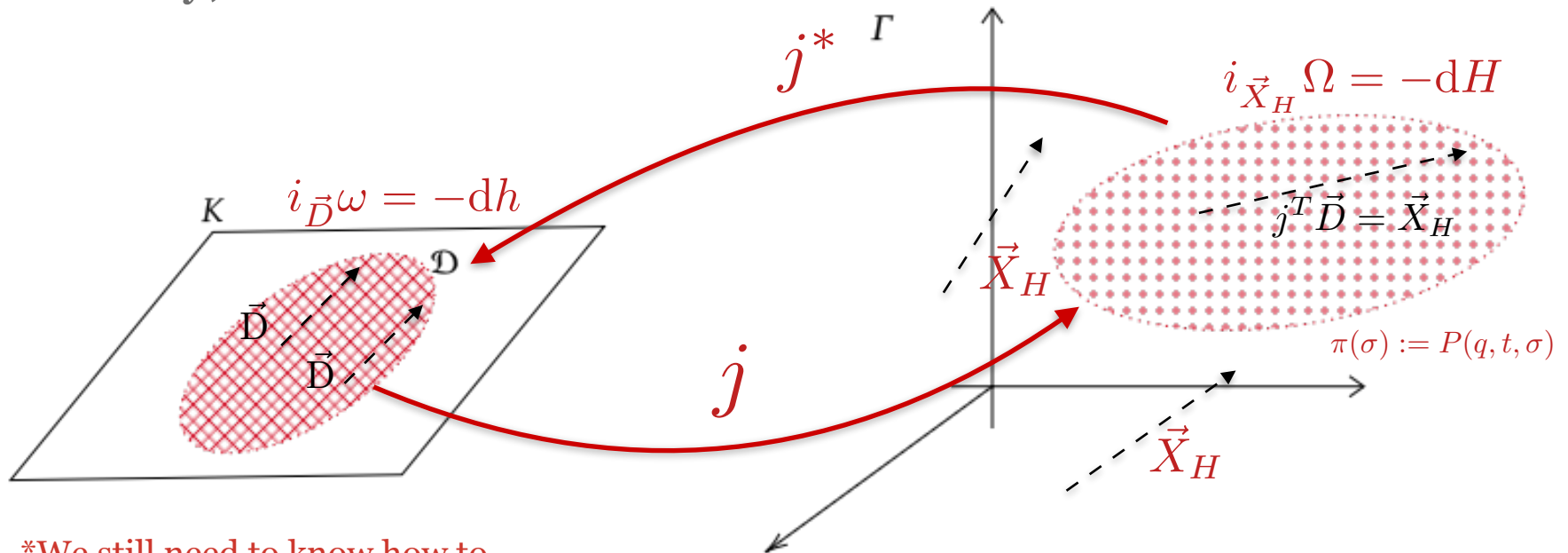
and, as $j^T \mathbf{D} = \mathbf{X}_H$, the Hamilton equations read

$$i_{\mathbf{X}_H} \Omega' = 0 \quad \xrightarrow{j^*} \quad i_{\mathbf{D}} \omega' = 0$$

Finally, the Hamiltonian in \mathcal{D}' is $h := j^* H$

$$h(q, t) = \int_{\mathbb{R}} d\sigma P(q, t, \sigma) \dot{q}(\sigma) - L(q, t)$$

Visually,



*We still need to know how to coordinate the dynamic space.

Applications

Consider the following nonlocal action:

$$K(\zeta) = e^{-|\zeta|}$$

$$S(\tilde{q}, R) = \int_{|t| \leq R} dt \left[\frac{1}{2} \dot{\tilde{q}}^2(t) - \frac{\omega^2}{2} \tilde{q}^2(t) + \frac{g}{4} \tilde{q}(t) \int_{\mathbb{R}} d\zeta K(\zeta) \tilde{q}(t - \zeta) \right]$$

Solution of **EOMs**: $q(\sigma) = \sum_{j=1}^4 B^j e^{\sigma r_j}$ with $|\operatorname{Re}(r_j)| < 1$

Stable: $g \leq 0$ or $\max\{1, 2\sqrt{g} - 1\} < \omega^2 < g$

These are the coordinates of \mathcal{D}'

These are the parametric equations of \mathcal{D}'

The **momenta**: $P(q, \sigma) = \sum_{j=1}^4 B^j \left(r_j \delta(\sigma) + \frac{g}{4} \frac{e^{-|\sigma|}}{r_j + \operatorname{sign}(\sigma)} \right)$

The elementary Poisson bracket:

The **symplectic form**: $\omega = \delta p_0 \wedge \delta q_0 + \delta \pi_0 \wedge \delta \xi_0$ $\{q_0, p_0\} = \{\xi_0, \pi_0\} = 1$

The **Hamiltonian**: $h(p_0, q_0, \pi_0, \xi_0) = \frac{1}{2} p_0^2 + \frac{\omega^2}{2} q_0^2 - \sqrt{g} q_0 \pi_0 + \frac{1}{2} \pi_0^2 - \frac{1}{2} \xi_0^2$

Dispersive Media:

$$M^{abcd}(x) = (2\pi)^{-2} \left[m(x) \hat{\eta}^{a[c} \hat{\eta}^{d]b} + 2\varepsilon(x) u^{[a} \hat{\eta}^{b][c} u^{d]} \right]$$

$$S(\tilde{A}, R) = \frac{1}{4} \int_{|x| \leq R} dx \tilde{F}_{ab}(x) \left(M^{abcd} * \tilde{F}_{cd} \right) (x)$$

The (**Belinfante-Rosenfeld**) energy momentum tensor:
(for a wave package)

$$\mathcal{U} \approx \frac{1}{4} \operatorname{Re} \left[\frac{d(\varepsilon\omega)}{d\omega} \underline{\tilde{\mathbf{E}}} \cdot \underline{\tilde{\mathbf{E}}}^* + \frac{\mu^*}{\mu} \frac{d(\mu\omega)}{d\omega} \underline{\tilde{\mathbf{H}}} \cdot \underline{\tilde{\mathbf{H}}}^* \right]$$

$$G^i \approx \frac{1}{4} \operatorname{Re} \left[\frac{1}{\omega\mu} \frac{d(\varepsilon\mu\omega^2)}{d\omega} \underline{\tilde{\mathbf{E}}} \times \underline{\tilde{\mathbf{B}}}^* \right]$$

$$S^i \approx \frac{1}{2} \operatorname{Re} \left[\underline{\tilde{\mathbf{E}}}^* \times \underline{\tilde{\mathbf{H}}}^* \right]$$

$$T^{ij} \approx \frac{1}{2} \operatorname{Re} \left[\underline{\tilde{E}}^{*i} \underline{\tilde{D}}^j + \underline{\tilde{H}}^i \underline{\tilde{B}}^{*j} - \frac{1}{2} \left(\underline{\tilde{\mathbf{D}}} \cdot \underline{\tilde{\mathbf{E}}}^* + \underline{\tilde{\mathbf{H}}} \cdot \underline{\tilde{\mathbf{B}}}^* \right) \delta^{ij} \right]$$

U(1) non-commutative gauge theory:

$$S(\tilde{A}, R) = \int_{|x| \leq R} dx \left(-\frac{1}{4} \tilde{F}^{ab}(x) \tilde{F}_{ab}(x) \right)$$

$$\tilde{F}_{ab} := \partial_a \tilde{A}_b - \partial_b \tilde{A}_a - i[\tilde{A}_a, \tilde{A}_b]$$

$$[f, g] = f * g - g * f, \quad (f * g)_{(x)} := \frac{1}{\pi^4 |\theta|} \int_{\mathbb{R}^8} du dv f(x-u) g(x-v) e^{2iu\tilde{\theta}v}$$

Second **Noether's theorem** :
(unexplained)

$$\left\{ \begin{array}{l} \delta \tilde{A}_a(x) = D_a \epsilon(x) \\ D_a f = \partial_a f - i[\tilde{A}_a, f] \end{array} \right.$$

$$D_a D_b \tilde{F}^{ab} \equiv 0$$

Conclusions

We have seen that:

- We develop a **new formalism** to deal with nonlocal Lagrangians.
- We **extend Noether's theorem** for them.
It gives us, by analogy with the local case, the definition of **the Legendre transform**.
- We propose a **Hamiltonian formalism** for them.
Once we know how to coordinate the extended dynamic space, it provides us with **the Hamiltonian** and **the symplectic form**.

Observations:

- All of the above we have **extended to classical field theory.**
- Some applications:
 - Nonlocal electrodynamics - **Dispersive Media.**
 - Nonlocal **harmonic oscillator.**
 - p-adic **strings.**
 - **Non-commutative** theories.

and, **gravity?**

Could this formalism be **applied**
to **nonlocal Gravity**?

$$\left. \begin{aligned} \text{IDG: } \mathcal{L} &\sim f(x) e^{-r\Box} f(x) \\ e^{-r\Box} &= \sum_{n=0}^{\infty} \frac{(-r\Box)^n}{n!} \quad \Box = \partial_{\mu}\partial^{\mu} \\ \text{GNL: } \mathcal{L} &\sim f(x) \frac{1}{\Box} f(x) \end{aligned} \right\} \sim f(x) \int_{\mathbb{R}^4} dy \overset{?}{K(x,y)} f(y)$$

*Is this formalism **coordinate-independent**?

Thank you!
Any question?

Prof. Josep Llosa



C. Heredia Pimienta Ph.D.



www.carlosherediapimienta.com