# FAQ Handbook: Common Questions, Helpful Answers

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**Abstract**

This document is designed to answer the questions I have been asked over my professional career about my work and research, providing a resource for anyone with similar inquiries to find the answers they seek.

## 1 Questions and Answers

### Q1: How does nonlocal Lagrangian formalism address the limitations of local theories in describing phenomena in quantum gravity and string theory?

Before addressing this question, it is crucial to acknowledge that the local Lagrangian formalism has been, and continues to be, one of the most effective frameworks developed to describe the dynamics of objects. However, there are certain"physical" limitations that need to be addressed, and one of the tools for tackling these limitations is non-locality.

The primary goal of the nonlocal Lagrangian formalism is to resolve: the problem of UV completion (singularities such as black holes in General Relativity or the Big Bang in cosmology). This issue can be summarized in one statement: *In physics, infinities do not exist.* Therefore, singularities, as they are currently understood, cannot exist, meaning that the divergences appearing in General Relativity or cosmological models must be interpreted differently.

This problem is, of course, addressed by models of quantum gravity, particularly string theory. In this model, interactions between strings are naturally non-point-like (unlike, say, interactions between billiard balls), but occur over a region, making them nonlocal [1]. Thus, there arises a need to rigorously construct a nonlocal Lagrangian formalism to describe the dynamics of these nonlocal phenomena.

### Q2: How does the nonlocal principle of least action differ from its local counterpart, and what implications does this difference have for the solutions of the resulting Euler-Lagrange equations?

In the local formalism, the principle of least action is based on finding those curves $q(t)$ that make the action stationary, i.e., $\delta S = 0$. It is well known that the $q(t)$ satisfying this condition are those that fulfill the Euler-Lagrange equations. Delving deeper into the process of deriving these equations, we realize that one of the steps involves integration by parts. The question now becomes clear: *Can we make the same step for nonlocal Lagrangians?* The answer is no. Nonlocal Lagrangians may not involve derivatives, which makes it impossible to make this step. Therefore, we must look for different alternatives to find the system's dynamic equations.

My doctoral thesis [2] proposes an alternative path to find these dynamic equations, and we can always recover to the local case if, instead of a nonlocal Lagrangian, we introduce a nonlocal one, thus confirming the extension of this principle.

The implication of obtaining the Euler-Lagrange equations by this procedure is that the operators appearing in the Lagrangian are not necessarily derivatives; they can be fractional derivatives, integrals with a specific kernel, etc. This has consequences for the Euler-Lagrange equations: now, the equations describing the dynamics do not necessarily have to be differential equations but can be integro-differential equations. This implies that the theorems of existence and uniqueness cannot be applied to ensure that the solution exists and is unique. Therefore, we cannot generally guarantee that the solutions to these integro-differential equations exist and are unique, unlike in the local case. Hence, we must consider and explore each case, system by system to ensure it [3].

## Q3: How could the nonlocal Lagrangian formalism contribute to the unification of gravity and quantum physics, specifically in the context of black holes and the early universe?

My PhD thesis is directed towards that goal, but, honestly, we are still quite far from achieving it. Quantum physics has been discovered and experimentally proven to be nonlocal (Bell's inequality [4]), but I believe more studies are needed to show how to relate it through non-locality in the Lagrangian formalism. The relationship is not yet clear. However, and this is one of the most interesting achievements of my research, we have taken the first step towards the quantization of a nonlocal theory by establishing the foundations of a nonlocal Hamiltonian formulation. Every quantum theory needs a Hamiltonian formulation to support it, and this is what we have developed.

The contribution of the nonlocal Lagrangian formalism in the context of black holes is addressed in Q1 (the UV completion problem) but is not exemplified for the early universe, and more specifically for Dark Energy. For example, in [5], it is proposed that the accelerated expansion of the universe could be due to nonlocal geometric effects, or in [6], cosmological solutions such as bouncing universes are obtained, which might avoid the Big Bang singularity problem. Therefore, the nonlocal formulation provides new solutions and physical aspects.

## Q4: What are the technical difficulties in formulating a Hamiltonian formalism for nonlocal Lagrangians, and how were these overcome in the thesis?

One of the achievements of my research is the rigorous definition of the Legendre transformation. The Legendre transformation is a mapping[1] that allows us (in simple terms) to relate the variables $(q, \dot{q})$ of the Lagrangian with the variables $(q, p)$ of the Hamiltonian. This transformation transfers the dynamics from the Lagrangian formulation to the Hamiltonian formulation.

In physics, the Legendre transform can be defined using Noether's theorem, which is a theorem posed on the Lagrangian formalism. This definition comes from the prefactors appearing in $\delta q$ of the boundary terms of this theorem. Delving into it, we observe that these prefactors are calculated by integration by parts. The question becomes clear again: *How can we integrate by parts a Lagrangian that does not contain derivatives, such as nonlocal Lagrangians?* The answer is: We cannot. We need to develop an alternative method to obtain it, and that is precisely what we have developed. We rigorously extended Noether's theorem to nonlocal Lagrangians, thereby proposing a well-defined Legendre transformation. This development enabled us to introduce a nonlocal Hamiltonian formulation, inclusive of its quantization.

It should be mentioned that this problem was addressed differently in the past. Without knowing how to proceed directly with a nonlocal Lagrangian, the approach was to transform it into an infinite-order one (i.e., with infinite derivatives) and apply Noether's theorem integrating by parts infinitely many times. This procedure was inspiring but lacked mathematical rigor since the convergence of these infinite series was not demonstrated, besides being tedious to handle them [7]. With our proposal, everything is simplified, compactified, and becomes more elegant, thus making it easier to capture and visualize the physics of these models.

Obviously, it goes without saying that this definition of the Legendre transform, if instead of using a nonlocal Lagrangian, we use a local Lagrangian, recovers the well-studied results from textbooks.

## Q5: What are the potential limitations of the nonlocal Lagrangian formalism as presented in the thesis, and how might future research address these challenges?

There are primarily two limitations. The first is the generalization of the Lagrangian and Hamiltonian formalism regardless of the coordinate system used. Currently (due to the brief time since its publication), we have not yet had the opportunity to generalize it for any coordinate system. This is crucial for gravity. Remember: Gravity is independent of the coordinate system used, so the Lagrangian formulation must be as well.

The second limitation is the existence and uniqueness theorem for integro-differential equations. At present, there is no general theorem that ensures the existence of a solution for these systems, nor one that guarantees their uniqueness. It would be an extremely significant advancement to obtain the necessary

[1] it is not necessarily bijective, for example in gauge theories it is not.

conditions to demonstrate that a solution exists and is unique. Currently, we can only assure this on a case-by-case basis.

These challenges will need to be tackled in the future, as, frankly, I currently do not know how to address them.

## Q6: Can the nonlocal Lagrangian formalism be integrated or complemented with existing models in theoretical physics, such as the Standard Model or General Relativity?

Yes, but they are understood as an extension of these theories[2], and some research groups dedicated to theoretical physics are currently working on this. In gravity, for example, [8] incorporates non-locality to Einstein's gravity through new terms that contain infinite-order derivatives, finding that the Newtonian potential $1/r$ does not diverge at $r = 0$, namely, it does not have singularities at linearized level. For instance, in [5], fractional derivatives, such as $1/\Box$, are incorporated to study accelerated models. More examples of gravity can be found in [9–11] and therein. For the Standard Model, there are generalized Maxwell models like [12]. In this model, the interaction of electromagnetic fields with media presenting dispersion is studied. In [13], nonlocal interaction amplitudes between particles are being investigated. Finally, for example, in [14], models for non-commutative (nonlocal) theories are studied. These models are based on the idea that coordinates cannot be permuted.

It is important to mention that, from a physical standpoint, one must proceed with caution when proposing nonlocal models. One of the main issues is causality. Non-local physical models suffer from this problem since they must take into account both the past and the future. This raises the question: *Can the future affect the past?*

## Q7: Can the nonlocal Lagrangian formalism be applied to Deep Learning?

As of today (20/03/24), we are working on it. So far, we have demonstrated that the continuous representation of the dynamics of adaptive optimization models, such as ADAM [15], can be represented with nonlocal Lagrangians, which fully opens the door of all my doctoral thesis work to the world of Deep Learning.

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